PHY140Y

Spring Term – Tutorial 20 Discussion Solutions

6 March 2000

1. (a) We can solve this by direct substitution into the left-hand side of equation (1) in the discussion question. We find that

$$\begin{aligned}
\frac{-\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+U(\vec{x})\psi &= \frac{-\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) && (1)\\
&= \frac{-\hbar^2}{2m}\left(-k_x^2-k_y^2\right)\psi && (2)\\
&= \frac{\hbar^2}{2m}\left(k_x^2+k_y^2\right)\psi && (3)\\
&= E\psi, && (4)
\end{aligned}$$

which means that ψ is a solution if we require that the total energy of the state be

$$E = \frac{\hbar^2}{2m}\left(k_x^2+k_y^2\right) \qquad (5)$$

(b) We apply the same boundary conditions as before, namely that $\psi = 0$ at $x = 0$ and L and at $y = 0$ and L. The proposed solution already satisfies the boundary conditions at $x = 0$ and $y = 0$. The boundary conditions at $x = L$ and $y = L$ are satisfied if

$$\begin{aligned}
\sin(k_x L) &= 0 && (6)\\
\Rightarrow k_x L &= n_x\pi \quad \text{where } n_x = 1, 2, 3, \ldots && (7)\\
\Rightarrow k_x &= \frac{n_x\pi}{L}, \quad n_x = 1, 2, 3, \ldots && (8)\\
\sin(k_y L) &= 0 && (9)\\
\Rightarrow k_y L &= n_y\pi \quad \text{where } n_y = 1, 2, 3, \ldots && (10)\\
\Rightarrow k_y &= \frac{n_y\pi}{L}, \quad n_y = 1, 2, 3, \ldots && (11)
\end{aligned}$$

(c) The three lowest lying energy states are

$$\begin{aligned}
E_0 &= \frac{\hbar^2\pi^2}{2mL^2}(1+1) = \frac{\hbar^2\pi^2}{mL^2} && (12)\\
E_1 &= \frac{\hbar^2\pi^2}{2mL^2}(1+4) = \frac{5\hbar^2\pi^2}{2mL^2} && (13)\\
E_2 &= \frac{\hbar^2\pi^2}{2mL^2}(4+4) = \frac{4\hbar^2\pi^2}{mL^2} && (14)\\
& && (15)
\end{aligned}$$

There is only one state with energy E_0 with $n_x = n_y = 1$. There are two states with energy E_1, given by $n_x = 1$ and $n_y = 2$ and $n_x = 2$ and $n_y = 1$. There is only one state with energy E_2: $n_x = n_y = 2$ (I leave the energy level diagram to the tutorial). Thus the degeneracies of these three states are 1, 2 and 1, respectively.

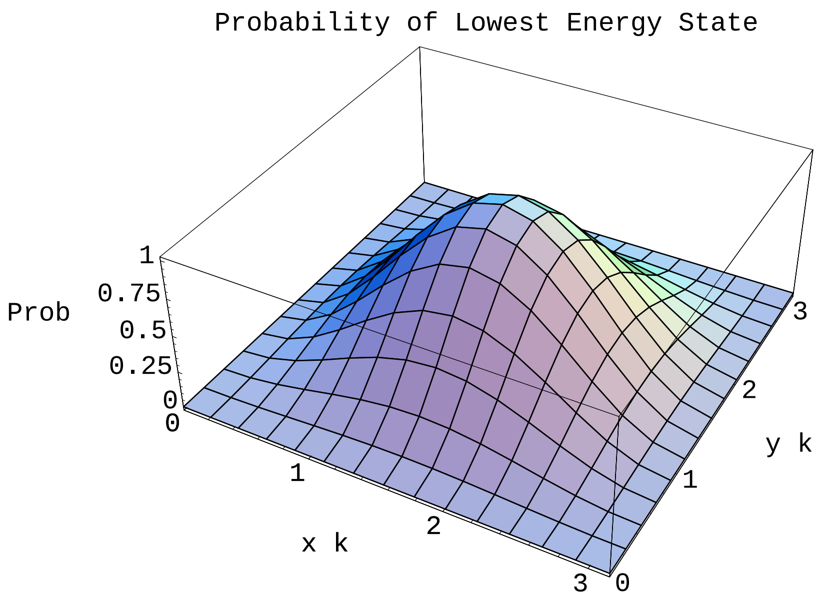

Figure 1: The probability distribution for the ground state of an electron in a 2-D box. I haven't bothered normalizing it (What is the normalization constant for this wave function?).

(d) In order for $E_0 = 10$ eV, we need

$$E_0 = \frac{\hbar^2\pi^2}{mL^2} \tag{16}$$

$$\Rightarrow L = \sqrt{\frac{\hbar^2\pi^2}{mE_0}} \tag{17}$$

$$= \sqrt{\frac{(6.58\times10^{-16})^2\pi^2}{(5.11\times10^6)(10)}} = 2.89\times10^{-19}\text{ m}, \tag{18}$$

where I have expressed $\hbar$ and the mass of the electron using units of electron Volts instead of Joules ($\hbar = 6.58\times10^{-16}$ eVs and $m = 5.11\times10^6$ eV).

(e) The probability distribution for the ground state is just $|\psi(x,y)|^2$ and would look like the plot shown in Fig. 1.

2. (a) In the region $x \in (0, L)$, we have

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + U(x)\psi_1 = \frac{\hbar^2k_1^2}{2m}\psi_1 \tag{19}$$

$$= E\psi_1 \tag{20}$$

$$\Rightarrow E = \frac{\hbar^2k_1^2}{2m} \tag{21}$$

$$\tag{22}$$

and for the region $x > L$, we have

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + U(x)\psi_2 = \frac{-\hbar^2k_2^2}{2m}\psi_2 + U_0\psi_2 \tag{23}$$

$$\begin{aligned} &= E\psi_2 && (24)\\ \Rightarrow U_0 - E &= \frac{\hbar^2 k_2^2}{2m}. && (25)\end{aligned}$$

Thus, ψ_1 and ψ_2 are solutions if the relationships between E, k_1 and k_2 are satisfied.

(b) We have already shown that

$$\begin{aligned} E &= \frac{\hbar^2 k_1^2}{2m} && (26)\\ U_0 - E &= \frac{\hbar^2 k_2^2}{2m}. && (27)\end{aligned}$$

(c) In order to tunnel as far as possible, we wish to make k_2 as small as possible (to reduce the variation in ψ_2 as a function of x). This can be achieved if we require that $U_0 - E$ be as small as possible.

The optimal value for L requires a little more thought. We have to consider the boundary conditions at $x = L$ to determine this. In this case, the boundary condition of continuity of the wave function and its derivative requires that we do some work. These conditions imply that

$$A\sin(k_1 L) = B\exp(k_2 L) \quad \text{and} \quad Ak_1\cos(k_1 L) = Bk_2\exp(k_2 L) \qquad (28)$$

$$\begin{aligned} \Rightarrow \sin(k_1 L) &= \frac{k_1}{k_2}\cos(k_1 L) && (29)\\ \Rightarrow \tan(k_1 L) &= \frac{k_1}{k_2}. && (30)\end{aligned}$$

What this means is that for a fixed ratio of k_1 and k_2 (which is determined by the relative sizes of E and U_0), L is fixed by this continuity condition, modulo a factor of $n\pi/k_1$ where n can be a positive integer. Since we are trying to find solutions that have ψ_2 as large as possible, the optimal choice for L is one that minimizes $k_2 L$ (the argument to the exponential in ψ_2), or the smallest value of L that also satisfies the condition in Eq. 30.

(d) We have already looked at the impact of the boundary conditions on k_1 and k_2. For given L and U_0, the condition in Eq. 30 determines the allowed energy levels. Once we have those, we then have to require that there be continuity at $x = l$, namely

$$\begin{aligned} A\sin(k_1 L) &= B\exp(k_2 L) && (31)\\ \Rightarrow B &= A\frac{\sin(k_1 L)}{\exp(k_2 L)} && (32)\end{aligned}$$

and that the entire wave function is normalized:

$$\begin{aligned} 1 &= \int_0^L |\psi_1^2|dx + \int_L^\infty |\psi_2^2|dx && (33)\\ &= \frac{A^2}{k_1}\int_0^{k_1 L}\sin^2 z dz + \frac{B^2}{k_2}\int_{2k_2 L}^\infty \exp(w)dw && (34)\end{aligned}$$

$$
\begin{aligned}
&= A^2 \left[\frac{1}{k_1} \int_0^{k_1 L} \sin^2 z dz + \frac{\sin^2(k_1 L)}{k_2 \exp(2k_2 L)} \int_{2k_2 L}^{\infty} \exp(w) dw \right] && (35) \\
\Rightarrow A &= \left[\frac{1}{k_1} \int_0^{k_1 L} \sin^2 z dz + \frac{\sin^2(k_1 L)}{k_2 \exp(2k_2 L)} \int_{2k_2 L}^{\infty} \exp(w) dw \right]^{-1/2}. && (36)
\end{aligned}
$$

3. The uncertainty in the electron's vertical position, Δy, leads to an uncertainty in the vertical component of their momentum, Δp_y, that must satisfy

$$
\Delta p_y \geq \frac{\hbar}{\Delta y}. \qquad (37)
$$

Since the electrons have a horizontal momentum of p_x, we can determine the minimum spread in the electron beam, $\Delta\theta$, by noting that

$$
\begin{aligned}
\tan \Delta\theta &= \frac{\Delta p_y}{p_x} && (38) \\
&\geq \frac{\hbar}{p_x \Delta y} && (39) \\
&= \frac{\hbar}{m_e v_x \Delta y} = \frac{1.055 \times 10^{-34}}{(9.11 \times 10^{-31})(2.2 \times 10^7)(9 \times 10^{-8})} && (40) \\
&= 5.85 \times 10^{-5} \text{ radians}. && (41)
\end{aligned}
$$

Note that the Heisenberg Uncertainty Principle applies to each coordinate separately!