PHY140Y

Solutions to Term Test 3

February 28, 2000

Section A (20%)

1. If the force of gravity has a $r^{-(2+\delta)}$ behaviour, then by equating the centripetal force required for circular motion with the gravitational force, we have the period $T = 2\pi r/v$ [1 mark] and

$$\frac{mv^2}{r} = \frac{GMm}{r^{(2+\delta)}} \quad (1)$$

$$\Rightarrow \frac{4\pi^2 r}{T^2} = \frac{GM}{r^{(2+\delta)}} \quad (2)$$

$$\Rightarrow T^2 = \frac{4\pi^2}{GM} r^{(3+\delta)} \quad (3)$$

   [2 marks]. So Kepler's Third Law is modified by having the power of r change by whatever value δ takes on [1 mark].

2. A spinning top is stable because it has significant angular momentum [2 marks] and it has very few if any external torques acting on it [1 mark]. Examples of rotating objects with few torques are frisbees and airplane gyroscopes [1 mark].

3. A tidal force is created when the force acting on an object varies with location [3 marks]. Such forces are **not** unique to gravity, as similar effects occur with any rapidly changing force field [1 mark].

4. The work function of a metal is the minimum energy that a photon must have to release an electron via the photoelectric effect [4 marks].

5. Compton scattering is the process by which a very high-energy photon scatters elastically off an electron [2 marks]. It was one of the clearest indications that photons interact with electrons in the same manner that other particles do, further illustrating the "particle" nature of the photon [2 marks].

Section B (80%)

1. Full marks on this question required you to write a coherent short essay addressing all or most of the issues outlined below, as well as providing an outline. Marks were awarded on content [8 marks], on organization and coherence [8 marks], and style [4 marks].

   (a) Classical physics was able to predict the experimental relationship between the power emitted by a glowing object and the temperature of the object (the Stefan-Boltzmann Law) and the relationship between the wavelength of maximum radiance and temperature (the Wein Displacement Law). The breakdown of the classical physics description of black body radiation had as its key component the fact that the observed power emitted by a black body as a function of wavelength was seen to fall off at lower wavelengths, whereas classical physics predicted that the power would rise rapidly as decreasing wavelength. This incorrect prediction was called the "ultraviolet catastrophe." This discrepancy then led to Planck's development of first his formula for the radiance of black body radiation as a function of wavelength and subsequently his theory explaining it, which predicted that the energy levels of an atom had to be discrete and the emitted photons had specific energies and wavelengths.

(b) The conservation of angular momentum is used in many systems to great advantage. One immediate example is the use of gyroscopes for navigational aids. A second example is the use of gyroscopes to allow spacecraft to adjust their attitude so as to maintain very stable orientations (the Hubble Space Telescope uses 6 gyroscopes in normal operation). Angular momentum is also used in large flywheels to maintain smooth motion under varying loads. This is the case in large generating systems and turbines. More prosaic examples where angular momentum is used is in a) figure-skating, where a skater is able to execute a fast spin by decreasing his or her moment of inertia, b) a playground merry-go-round, where the rotating deck is put into rotational motion with children riding on it, and c) the toss of a frisbee. There are many other possible examples!

2. (a) The torque acting on the molecule about the axis through its centre point is

$$\begin{aligned} \vec{\tau} &= \sum_i \vec{r}_i \times \vec{F}_i & (4) \\ &= -Q_+ E\frac{l}{2}\sin(\pi-\theta)\hat{P} + Q_- E\frac{l}{2}\sin(\pi-\theta)\hat{P} & (5) \\ &= \frac{-\left(Q_+ - Q_-\right)El}{2}\sin\theta\ \hat{P} & (6) \end{aligned}$$

[7 marks] where the axis $\hat{P}$ points out of the page.

(b) There will be an equilibrium point at $\theta = 0$, and it will oscillate about this point. The equation of motion will be given by

$$\vec{\tau} = I\vec{\alpha}, \quad (7)$$

where the moment of inertia I is just that of a rod about an axis through its centre, namely

$$I = \frac{1}{12}ml^2. \quad (8)$$

[6 marks] The equation of motion is thus

$$\begin{aligned} \frac{-\left(Q_+ - Q_-\right)El}{2}\sin\theta &= \frac{1}{12}ml^2\frac{d^2\theta}{dt^2} & (9) \\ \Rightarrow \frac{d^2\theta}{dt^2} &= \frac{-6\left(Q_+ - Q_-\right)E}{ml}\sin\theta. & (10) \end{aligned}$$

(c) From b), we find that for small oscillations ($\theta \ll 1$)

$$\frac{d^2\theta}{dt^2} \simeq \frac{-6\,(Q_+ - Q_-)\,E}{ml}\,\theta \tag{11}$$

$$\Rightarrow \theta(t) = A\sin(\omega t) + B\cos(\omega t) \tag{12}$$

[3 marks] for the general case. Requiring that the boundary conditions be satisfied,

$$\theta(0) = \theta_0 \text{ and } \frac{d\theta}{dt}(0) = 0, \tag{13}$$

$$\Rightarrow \theta(t) = \theta_0 \cos(\omega t) \tag{14}$$

where the angular frequency ω is [3 marks]

$$\omega = \sqrt{\frac{6\,(Q_+ - Q_-)\,E}{ml}} \tag{15}$$

$$= \sqrt{\frac{6(1\times 10^{-17})(2\times 10^{6})}{(2\times 10^{-25})(2\times 10^{-8})}} = 3\times 10^{22}\ \text{s}. \tag{16}$$

(d) The oscillatory motion will still continue in the rest frame of the molecule, but the molecule itself will accelerate in the direction opposite to the electric field. This is because there is a net force on the molecule of

$$\vec{F} = Q_+\vec{E} + Q_-\vec{E} \tag{17}$$

$$= (Q_+ + Q_-)\,\vec{E} = m\frac{d^2x}{dt^2}\,\hat{x}, \tag{18}$$

[4 marks] where x is in the direction of the electric field. The solution to this is

$$x(t) = \frac{(Q_+ + Q_-)\,t^2}{2m}\,E \tag{19}$$

$$= -(5\times 10^{13})t^2\ \text{m}, \tag{20}$$

which is a very rapid acceleration.

3. (a) The wavelength, λ_{max}, of the photon that will generate a photoelectron when the work function $\phi = 2.1$ eV is determined by the minimum photon energy, E_γ:

$$E_\gamma = \phi = \frac{hc}{\lambda_{max}} \tag{21}$$

$$\Rightarrow \lambda_{max} = \frac{hc}{\phi} \tag{22}$$

$$= \frac{(6.63\times 10^{-34})(3\times 10^{8})}{(2.1)(1.602\times 10^{-19})} = 5.9\times 10^{-7}\ \text{m}. \tag{23}$$

[7 marks]

(b) The energy of the electron as it is released from the surface, E_s, is

$$E_s = E_\gamma - \phi \quad (24)$$
$$= \frac{hc}{\lambda} - \phi \quad (25)$$
$$= \frac{(6.63 \times 10^{-34})(3 \times 10^8)}{(300)(1.602 \times 10^{-19})} - 2.1 = 2.0 \text{ eV}. \quad (26)$$

[7 marks] It is then accelerated through a potential difference of 1×10^3 V, giving it an energy of 1×10^3 eV. Thus, the electron's energy after acceleration is

$$E_f = E_s + (1 \times 10^3) = 1{,}002 \text{ eV}. \quad (27)$$

(c) The current flow is the number of Coulombs per second. This is equal to the rate of electrons being produced times the charge per electron,

[6 marks]

$$I = R_p(20)(1.602 \times 10^{-19}) = 3.2 \times 10^{-12} \text{ A}. \quad (28)$$

4. (a) The ball is thrown with a velocity $v_i = 160$ km/h $= 44.44$ m/s, at an angle θ from the horizontal. The subsequent horizontal and vertical positions of the ball will be

$$x(t) = v_i \cos\theta \; t \quad (29)$$
$$y(t) = v_i \sin\theta \; t - \frac{1}{2}\, g t^2. \quad (30)$$

If Δt is the time taken to cover the distance $D = 20$ m, then the vertical position and speed at $\Delta t/2$ will satisfy

$$y_{max} = v_i \sin\theta \; \Delta t/2 - \frac{1}{2}\, g(\Delta t/2)^2 \quad (31)$$
$$v_y = 0 = v_i \sin\theta - g(\Delta t/2). \quad (32)$$

[6 marks] We also know that

$$D = v_i \cos\theta \; \Delta t \quad (33)$$
$$\Rightarrow \Delta t = \frac{D}{v_i \cos\theta}. \quad (34)$$

We can substitute that expression into Eq. 32 to eliminate Δt:

$$0 = v_i \sin\theta - \frac{gD}{v_i \cos\theta} \quad (35)$$
$$\Rightarrow v_i^2 \, 2\sin\theta \, \cos\theta = gD \quad (36)$$
$$\Rightarrow v_i^2 \, \sin(2\theta) = gD \quad (37)$$
$$\Rightarrow \theta = \frac{1}{2}\sin^{-1}\left(\frac{gD}{v_i^2}\right) \quad (38)$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{(9.8)(20)}{(44.44)^2}\right) = 0.050. \quad (39)$$

We can then solve for the time taken,

$$\Delta t = \frac{D}{v_i \cos\theta} = 0.451 \text{ s.} \tag{40}$$

Then, we can subsitute these values back into the expression for the maximum deflection

$$y_{max} = v_i \sin\theta \, \Delta t/2 - \frac{1}{2} \, g(\Delta t/2)^2 \tag{41}$$

$$= (44.44)(0.050)(0.451/2) - \frac{1}{8}(9.8)(0.451)^2 = 0.25 \text{ m.} \tag{42}$$

(b) Uniform circular motion requires that the force F acting on the ball with mass m must satisfy

$$F = \frac{mv^2}{r}, \tag{43}$$

[6 marks]

where v is the speed of the ball and r is the radius of its orbit. This must equal the gravitational attraction of the asteroid, whose mass M is given by

$$M = \frac{4}{3} \pi r^3 \rho. \tag{44}$$

Thus, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \tag{45}$$

$$\Rightarrow \frac{mv^2}{r} = \frac{4\pi G \rho m r}{3} \tag{46}$$

$$\Rightarrow r = \sqrt{\frac{3v^2}{4\pi\rho G}} \tag{47}$$

$$= \sqrt{\frac{3(44.44)^2}{4\pi(2.5 \times 10^3)(6.67 \times 10^{-11})}} = 5.32 \times 10^4 \text{ m.} \tag{48}$$

(c) In this case, the energy of the ball with mass m is

[3 marks]

$$E = \frac{-GMm}{2r} \tag{49}$$

$$= \frac{-4\pi G \rho r^2 m}{6}. \tag{50}$$

(d) The speed of the ball would have to equal its escape velocity, which is the speed that makes the kinetic energy equal to the negative of the potential energy of the ball, ie.

[3 marks]

$$\frac{1}{2} m v_{es}^2 = \frac{GMm}{r} \tag{51}$$

$$\Rightarrow v_{es} = \sqrt{\frac{2GM}{r}} \tag{52}$$

$$= \sqrt{\frac{8\pi G\rho r^2}{3}} \tag{53}$$

$$= \sqrt{\frac{8\pi(6.67\times10^{-11})(2.5\times10^3)(3.32\times10^4)^2}{3}} = 62.88 \text{ m/s.} \tag{54}$$

This is a fairly well hit ball, and shows you must be careful when playing ball on an asteroid (or bring lots of balls).

(e) Even if the ball is hit with this speed at a 45° angle from the vertical, it still has enough kinetic energy to escape the gravitational pull of the asteroid. However, it's orbit in this case would be a parabola.

[2 marks]

5. (a) The motion of the skaters is circular rotation about a common centre of mass (which is at rest). The initial angular velocity, ω_i, of a skater at the point they grasp the bar is the same as their linear velocity, v, so that

$$\omega_i = \frac{v}{r_i} \tag{55}$$

$$= \frac{10}{1.5} = 6.67 \text{ rad/s,} \tag{56}$$

[7 marks] where r_i is the radius of the initial circular motion (half the length of the bar).

(b) By reducing their separation by a factor of 3, they have reduced their moment of inertia by a factor

$$\frac{I_f}{I_i} = \frac{2m_s r_f^2}{2m_s r_i^2} = \frac{1}{9}, \tag{57}$$

[6 marks] where I_i and I_f are the initial and final moments of inertia, m_s is the mass of each skater and r_f is the final radius of rotation. Since angular momentum is conserved, the final angular velocity must satisfy

$$I_i\omega_i = I_f\omega_f \tag{58}$$

$$\Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i \tag{59}$$

$$= 9\omega_i = 60 \text{ rad/s.} \tag{60}$$

(c) In part a), the kinetic energy is

$$K_i = \frac{1}{2}I_i\omega_i^2 \tag{61}$$

$$= m_s r_i^2\omega_i^2 \tag{62}$$

$$= (50)(1.5)^2(6.67)^2 = 5.0\times10^3 \text{ J.} \tag{63}$$

[7 marks] Similarily, the final kinetic energy is

$$\begin{aligned} K_f &= \frac{1}{2} I_f \omega_f^2 && (64) \\ &= m_s r_f^2 \omega_f^2 && (65) \\ &= (50)(0.5)^2(60)^2 = 4.5 \times 10^4 \text{ J}. && (66) \end{aligned}$$

6. (a) The volume of the wire of length $l = 2$ m is

$$V_w = \pi r_w^2 l \qquad (67)$$

so the charge density in the wire is

$$\begin{aligned} \rho_w &= \frac{Q_w}{V_w} && (68) \\ &= \frac{2.0 \times 10^{-9}}{\pi(5 \times 10^{-4})^2(2)} = 1.27 \times 10^{-3} \text{ C/m}^3. && (69) \end{aligned}$$

[6 marks] The volume of the metal sheath surrounding the insulator is

$$V_s = 2\pi r_s r_m l, \qquad (70)$$

where we have approximated this by calculating the surface area of the sheath and multiplying by its thickness. The charge density on the sheath is

$$\begin{aligned} \rho_s &= \frac{Q_s}{V_s} && (71) \\ &= \frac{-2.0 \times 10^{-9}}{2\pi(5 \times 10^{-3})(5 \times 10^{-4})(2)} = -6.37 \times 10^{-5} \text{ C/m}^3. && (72) \end{aligned}$$

(b) Gauss's Law states that the flux through a cylinder of radius r and length L surrounding the wire and inside the insulator is proportional to the charge enclosed. This implies that

[6 marks]

$$\begin{aligned} \int_{cyl} \vec{E}(r) \cdot \hat{n}\, dS &= 2\pi r E(r) L && (73) \\ &= \frac{1}{\epsilon_0} \int_{vol} \rho\, dV && (74) \\ &= \frac{\pi r_w^2 L \rho_w}{\epsilon_0} && (75) \\ \Rightarrow E(r) &= \frac{r_w^2 \rho_w}{2\epsilon_0 r}. && (76) \end{aligned}$$

(c) The potential difference, ΔV, between the wire and the outer sheath is given by the integral of the electric field between $r = r_w$ and $r = r_s$, namely

[4 marks]

$$\Delta V = \int_{r_w}^{r_s} E(r) dr \tag{77}$$

$$= \frac{r_w^2 \rho_w}{2\epsilon_0} \int_{r_w}^{r_s} \frac{1}{r} dr \tag{78}$$

$$= \frac{r_w^2 \rho_w}{2\epsilon_0} \ln\left(r_s/r_w\right) \tag{79}$$

$$= \frac{(5 \times 10^{-4})^2 (1.27 \times 10^{-3})}{(2)(8.85 \times 10^{-11})} \ln(10) = 4.13 \text{ V.} \tag{80}$$

(d) The capacitance per unit length of the cable is the charge it carries per unit length divided by the voltage difference:

[4 marks]

$$C \equiv \frac{Q_w/l}{\Delta V} \tag{81}$$

$$= \frac{2.0 \times 10^{-9}}{(2)(4.13)} = 2.42 \times 10^{-10} \text{ F.} \tag{82}$$