PHY140Y

11 Parallell Plate Capacitor and Angular Motion

11.1 Overview

- Parallel Plate Capacitor
- Rigid Body Rotation
- Angular Variables

11.2 Parallel Plate Capacitor

The parallel plate capacitor consists of two identical plates of area A that are held a distance L apart, as shown in Fig. 1. If these are conducting plates (which we can assume they are), and we connect them together in such a way that there is a potential difference between the plates (eg., this could be done with a 1.5 V battery), then the top plate will accumulate a charge Q, and the bottom one will accumulate a charge $-Q$. We now have enough information to determine how the charge on the plates (which is why a capacitor is considered as a device that stores charge) is related to the potential difference across the plates.

Let's first consider the top plate, with area A and charge Q. Let's define our coordinate system so that the $\hat{z}$ axis is normal to the plates. We will place the bottom plate at $z = 0$ and place the top plate at $z = L$. The surface charge density on the top plate will be

$$\sigma_t \equiv \frac{Q}{A}. \tag{1}$$

From the previous lecture, we know that the electric field associated with this plate will be pointing away from the plate and will be

$$\vec{E}_t = \begin{cases} \sigma_t/(2\epsilon_0)\hat{z}, & z > L, \\ -\sigma_t/(2\epsilon_0)\hat{z}, & z < L. \end{cases} \tag{2}$$

For the bottom plate, a similar analysis shows that the electric field is now pointing towards the plate and will be

$$\vec{E}_b = \begin{cases} -\sigma_t/(2\epsilon_0)\hat{z}, & z > 0, \\ \sigma_t/(2\epsilon_0)\hat{z}, & z < 0. \end{cases} \tag{3}$$

Note that the charge density on the bottom plate is $-\sigma_t$, and this has been taken into account in Eq. 3. The total field of the two parallel plates is, by superposition, the sum of the individual fields. This sum yields

$$\vec{E}_{tot} = \begin{cases} 0, & z > L, \\ -\sigma_t/(\epsilon_0)\hat{z}, & 0 < z < L, \\ 0, & z < 0. \end{cases} \tag{4}$$

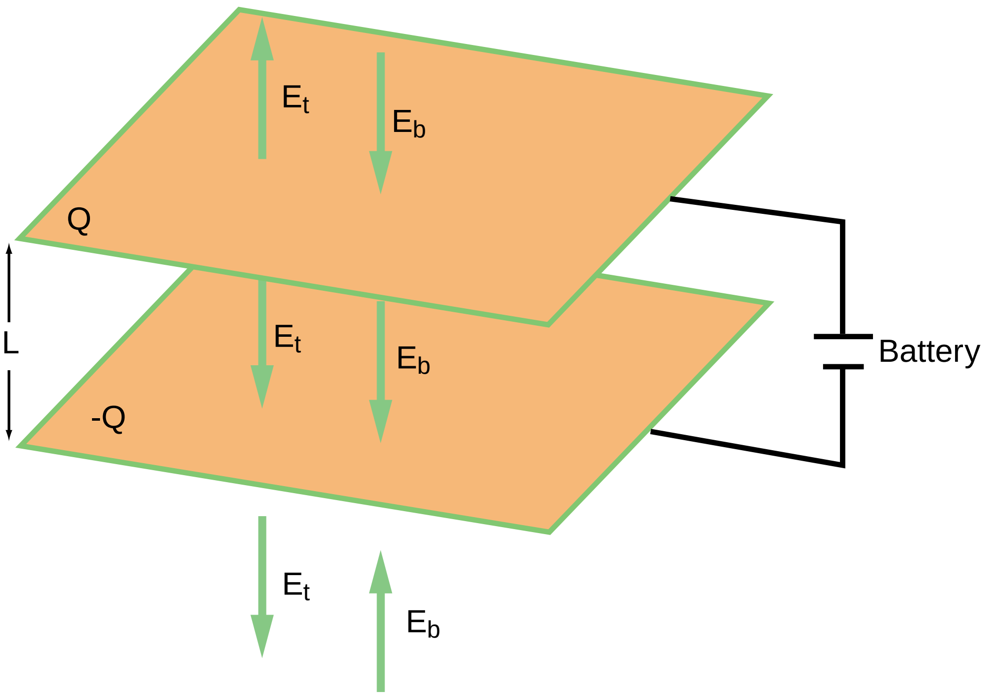

Figure 1: A parallel plate capacitor consisting of two plates of area A, the top one having a charge $+Q$ uniformly distributed on its surface and the bottom one holding a charge $-Q$ uniformly distributed on its surface.

This tells us that the electric field is zero except between the plates where it is a constant value, pointing downwards in the $\hat{z}$ direction.

Now, we want to determine the potential difference between the two plates, $\Delta V = V_{top} - V_{bottom}$. The work performed by the electric field when we move a test charge q_t from the bottom plate to the top plate is equal to $-\Delta V/q_t$, from our definition of the electric potential. Thus,

$$\begin{aligned} \Delta V &= -W/q_t && (5) \\ &= \frac{-q_t \vec{E}_{tot} \cdot L\hat{z}}{q_t} && (6) \\ &= (\sigma_t/\epsilon_0)\hat{z} \cdot L\hat{z} && (7) \\ &= \frac{QL}{\epsilon_0 A}, && (8) \end{aligned}$$

where we have reintroduced the total charge and area of the top plate.

This shows us that the total charge stored on each plate will be proportional to the voltage difference placed across the plate. When we hook up a battery to the plates, where does the charge come from? It is provided by the battery itself, draining some of the charge stored in the battery.

11.2.1 Capacitance

We often characterize a capacitor by its **capacitance** C, defined as

$$C \equiv \frac{Q}{\Delta V}, \tag{9}$$

which has units of charge/voltage or Coulombs/Volt. The SI unit for this quantity is the Farad. For a parallel plate capacitor, the capacitance becomes

$$\begin{aligned} C &= \frac{Q}{\Delta V} && (10) \\ &= \frac{\epsilon_0 A}{L}. && (11) \end{aligned}$$

To give some idea of what a typical capacitor would be like, let's assume that we have a parallel plate capacitor with plates 2 mm on a side, separated by 1 mm. The capacitance of this arrangement is

$$\begin{aligned} C &= \frac{\epsilon_0 A}{L} && (12) \\ &= \frac{(8.85 \times 10^{-11})(4 \times 10^{-6})}{1 \times 10^{-3}} = 3.54 \times 10^{-14} \text{ F}. && (13) \end{aligned}$$

We usually deal with small capacitances, so the typical unit used is the picoFarad, or 10^{-12} Farad (often called a "puff"). This capacitor would therefore have a capacitance of 0.035 pF.

11.3 Rigid Body Rotation

We have up till now only considered kinematics in a context where we talk about the displacement of point masses or centres of mass and apply Newton's Laws. We therefore have assumed that the motion of masses are unconstrained. We are now going to look at a particular type of motion where the body in motion is constrained to keep one point fixed in space, so that the only allowed motion is rotation about this point. We will call this "rigid-body rotation" and spend the next few lectures analyzing this specialized type of motion.

The generic picture is illustrated in Fig. 2. A rigid body rotates about an axis $\hat{P}$. We usually define a fixed reference axis perpendicular to $\hat{P}$ and then identify the position of a point $\vec{r}(t)$ by the rotation angle θ and its perpendicular distance from the axis of rotation, r (which is fixed).

11.3.1 Angular Velocity

Having identified a rotation axis $\hat{P}$ about which a rigid body will rotate, we can then describe the orientation of this rigid body by a single angle θ. If in a time interval Δt, the body rotates through an angle $\Delta\theta$, then we can define the average angular velocity as

$$\bar{\omega} \equiv \frac{\Delta\theta}{\Delta t}, \tag{14}$$

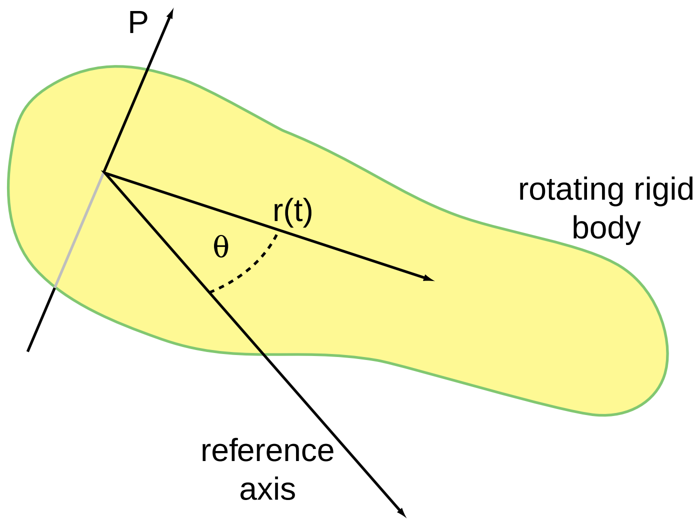

Figure 2: A generic rigid body rotation about an axis $\hat{P}$. A reference axis is defined about which

where the units of $\bar{\omega}$ are rad/s (ω is the Greek letter "omega"). For a given point in the rigid body, the average velocity of this point would be $\bar{v} = r\bar{\omega}$, where r is the perpendicular distance the point is away from the axis. We define the sign of $\Delta\theta$ using the right-hand rule. A rotation about the axis $\hat{P}$ that is in the same direction that the fingers of your right hand point in when your thumb is aligned with $\hat{P}$ will be a **positive** rotation.

If we allow the time interval Δt shrink to zero, then in this limit we can define the magnitude of the instantaneous angular velocity as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \tag{15}$$

$$= \frac{d\theta}{dt}. \tag{16}$$

By definition, we define the instantaneous angular velocity as a vector with direction parallel to the $\hat{P}$ and magnitude as given above. Thus,

$$\vec{\omega} = \omega\hat{P}. \tag{17}$$

As a concrete example, we can consider the Earth as a rigid-body rotating uniformly about an axis $\hat{P}$ running through the north and south poles. The instantaneous angular velocity is

$$\omega = \frac{2\pi}{1 \text{ day}} \tag{18}$$

$$= \frac{2\pi}{8.64 \times 10^4} = 7.27 \times 10^{-5} \text{ rad/s} \tag{19}$$

with a direction pointing up through the north pole (check this – where does the sun rise and set?).

11.3.2 Angular Acceleration

We can now define the average angular acceleration as

$$\bar{\alpha} \equiv \frac{\Delta\omega}{\Delta t}, \tag{20}$$

where $\Delta\omega$ is the change in the instantaneous angular velocity in a time interval Δt (α is the Greek letter "alpha"). In the limit where Δt shrinks to zero, we define the magnitude of the instantaneous angular acceleration

$$\begin{aligned} \alpha &= \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} && (21) \\ &= \frac{d\omega}{dt} && (22) \\ &= \frac{d^2\theta}{dt^2}. && (23) \end{aligned}$$

We define the instantaneous angular acceleration as a vector with direction parllel to $\hat{P}$:

$$\vec{\alpha} = \alpha\hat{P}. \tag{24}$$