PHY140Y

12 Torque and Moments of Inertia

12.1 Overview

- Torque
- Moments of inertia
- Example: Bicycle wheel

12.2 Torque

For a object rotating about a fixed axis $\hat{P}$, we have defined the angular displacement $\theta(t)$ relative to some fixed reference axis, the angular velocity $\vec{\omega}(t)$ and the angular acceleration $\vec{\alpha}(t)$. These are the rotational analogues of the displacement $\vec{x}(t)$, velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$. What is the analogue of Newton's Second Law of Motion?

Let's look at the behaviour of a simple pendulum formed by a mass M attached to a massless rod that swings about an axis $\hat{P}$ located at one end, as shown in Fig. 1. If we apply a force $\vec{F}_{\|}$ to the mass that is parallel to the arm, we intuitively see that the mass will not move. In fact, we know that the tension exerted by the arm will act in an equal and opposite manner to keep the mass motionless. However, if we take the same force and orient it so that it is at right angles to the rod, we would expect the mass to accelerate and begin to rotate under this new force $\vec{F}_{\perp}$. In fact, intuitively, we would expect the angular motion of the mass to accelerate, ie., $\vec{\alpha} \neq 0$. So this shows us that only the component of the force perpendicular to the vector defining the point where the force is applied relative to the axis $\hat{P}$ matters.

It is therefore convenient to define the **torque** as

$$\vec{\tau} \equiv \vec{r} \times \vec{F}, \tag{1}$$

ie, $\vec{\tau}$ is the cross-product of the vector defining the location of the force and the force itself. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$, and is given by the "right-hand rule" – align your fingers of your right-hand along the direction needed to redirect $\vec{r}$ to get it to line up with $\vec{F}$ and your thumb then points in the direction of $\vec{\tau}$. It's magnitude is given by

$$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\psi, \tag{2}$$

where ψ is the angle between $\vec{r}$ and $\vec{F}$.

Let's see what Newton's Second Law tells us about the motion in terms of this torque. If $\vec{F}$ is the applied force on the mass m, from our discussion above we can write

$$\vec{F} \sin\psi = m\vec{a} \tag{3}$$

$$\Rightarrow F \sin\psi = m\frac{\Delta v}{\Delta t} \tag{4}$$

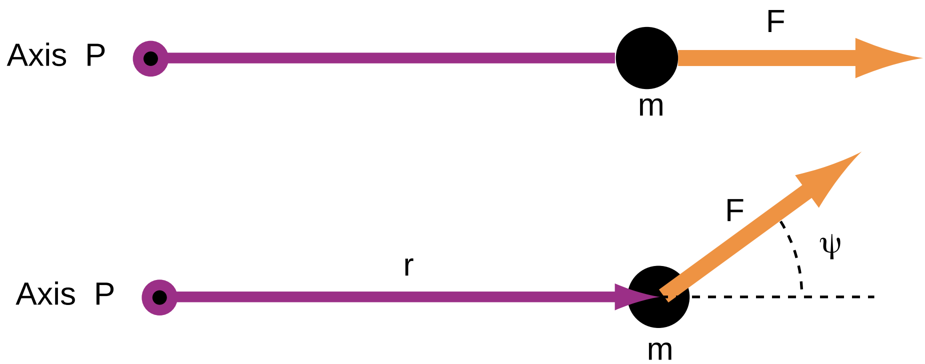

Figure 1: Two possible orientations of a force $\vec{F}$ acting on a simple pendulum consisting of a massless rod and connecting a mass m to a pivot axis $\hat{P}$.

$$\begin{aligned} &\simeq mr\frac{\Delta\omega}{\Delta t} && (5)\\ \Rightarrow Fr\sin\psi &= mr^2\frac{\Delta\omega}{\Delta t}, && (6)\end{aligned}$$

where $\vec{a}$ is the instantaneous linear acceleration of the mass m. We will call the quantity mr^2 the **moment of inertia** of the mass m relative to the axis $\hat{P}$. Conventionally, we will denote it by the letter I and as its definition implies it has units of $\mathrm{kg\,m}^2$. With this definition, our equations of instantaneous motion become

$$\begin{aligned} Fr\sin\psi &= mr^2\frac{\Delta\omega}{\Delta t} && (7)\\ \Rightarrow \vec{\tau} &= I\vec{\alpha}, && (8)\end{aligned}$$

where we note that both $\vec{\tau}$ and $\vec{\alpha}$ have directions along $\hat{P}$. This is the rotational analogue of Newton's Second Law, where we make the correspondence

$$\begin{aligned} \vec{\tau} &\Leftrightarrow \vec{F} && (9)\\ I &\Leftrightarrow m && (10)\\ \vec{\alpha} &\Leftrightarrow \vec{a} && (11)\\ \vec{\omega} &\Leftrightarrow \vec{v} && (12)\\ \theta &\Leftrightarrow \vec{x}. && (13)\end{aligned}$$

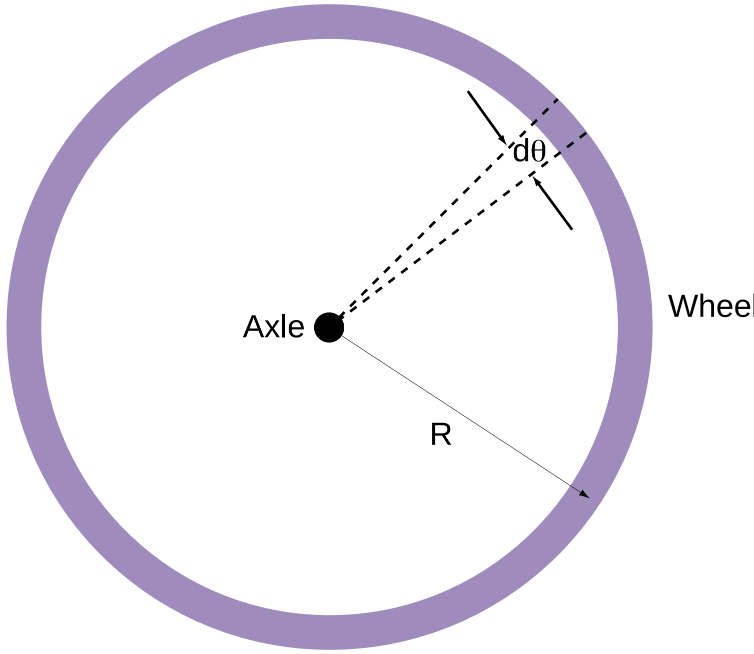

Figure 2: The bicycle wheel.

12.3 Moments of Inertia

The moment of inertia concept, introduced in the context of a small mass at the end of a rod, can be extended to distributed masses where the object has a volume V and a mass density $\rho(\vec{r})$. It is a straightforward exercise to show that for such an object the moment of inertia about an axis $\hat{P}$ is given by

$$I = \int_V r^2 \rho(\vec{r}) dV, \tag{14}$$

where $\vec{r}$ is the position vector defined with respect to the location of the axis $\hat{P}$. Note that the quantity $\rho(\vec{r})dV$ is just the mass dM located in the volume element dV.

I emphasize (and we will show below) that the moment of inertia is always defined with respect to a given rotation axis. If the rotation axis changes, the moment of inertia will also change. So let's do a few examples of calculations of moments of inertia.

12.3.1 Bicycle Wheel

Suppose we have a bicycle wheel, with a mass $M_w = 1.9$ kg and a radius of $R = 32$ cm, as shown in Fig. 2. Let us calculate the moment of inertia of this wheel about its axle.

We first note that, provided the wheel is thin enough, we can assume that all the mass is a distance $r = R$ away from the axle. With this observation, one can conclude that the actual moment of inertia is just

$$\begin{aligned} I_w &= R^2 M_w & (15) \\ &= (0.32)^2(1.9) = 0.195 \text{ kg m}^2! & (16) \end{aligned}$$

Whoa! Getting the answer this fast is not fair. Let's do this more carefully, taking the general definition of the moment of inertia we wrote down above. We need to perform a volume integral over the wheel, so to do that, let's break up the wheel into small segments defined by an angular interval $d\theta$ with respect to axle, as shown in Fig. 2. Since the entire circumference of the wheel is $C = 2\pi R$, the mass associated with the angular interval $d\theta$ is

$$\begin{aligned} dM &= M_w \frac{R\, d\theta}{2\pi R} && (17) \\ &= M_w \frac{M_w}{2\pi}\, d\theta && (18) \end{aligned}$$

which is the specific form for the mass of the volume element defined by $d\theta$. Plugging this form into volume integral from Eq. 14, we get

$$\begin{aligned} I_w &= \int_V r^2 \rho(\vec{r}) dV && (19) \\ &= \int_0^{2\pi} \frac{R^2 M_w}{2\pi}\, d\theta && (20) \\ &= \frac{R^2 M_w}{2\pi} \int_0^{2\pi} d\theta && (21) \\ &= R^2 M_w. && (22) \end{aligned}$$

So the quick way was right, after all.