PHY140Y

13 More on Torques and Moments of Inertia

13.1 Overview

- Torque on a bicycle wheel
- More moments of inertia

13.2 Torque on Bicycle Wheel

Let's look at the bicycle wheel we had considered in the last lecture, and now suppose that there is a valve stem on the wheel with a mass $m_v = 25$ g. The gravitational force on the valve stem is going to cause the wheel to rotate. Let's see how we can understand this in the context of our angular laws of motion.

Let the angular displacement of the valve stem from it's most stable position (where the stem as at the bottom of the wheel) be given by the angle θ, as shown in Fig. 1. First let's work out the new moment of inertia of the wheel, including the valve stem. That will just be the original moment of inertia plus the moment of inertia of the valve stem itself relative to the wheel's axle. Therefore,

$$\begin{aligned} I'_w &= I_w + R^2 m_v & (1) \\ &= 0.197\ \mathrm{kg\,m^2}. & (2) \end{aligned}$$

How suppose initially the wheel is at rest with the valve stem located at an angle θ_0. The force on the valve stem will be the gravitational force $\vec{F}_g$, as shown in Fig. 1. The torque that is associated with this force when the angular displacement is θ will be

$$\begin{aligned} \vec{\tau}_w &= \vec{r} \times \vec{F}_g & (3) \\ &= -R\,(m_v g) \sin\theta\, \hat{P}, & (4) \end{aligned}$$

where we have explicitly calculated the magnitude of the torque. The vector $\hat{P}$ is along the axle pointing out of the page. We can relate this to the angular acceleration:

$$\begin{aligned} \vec{\tau}_w &= I'_w \vec{\alpha} & (5) \\ \Rightarrow -R\,(m_v g) \sin\theta\, \hat{P} &= I'_w \vec{\alpha} & (6) \\ \Rightarrow \vec{\alpha} &= \frac{-R m_v g}{I'_w} \sin\theta\, \hat{P}. & (7) \end{aligned}$$

We see this is a restoring torque, as we would expect. If we displace the valve from its equilibrium position, we would expect the motion to be periodic and oscillatory. If the displacement θ is small

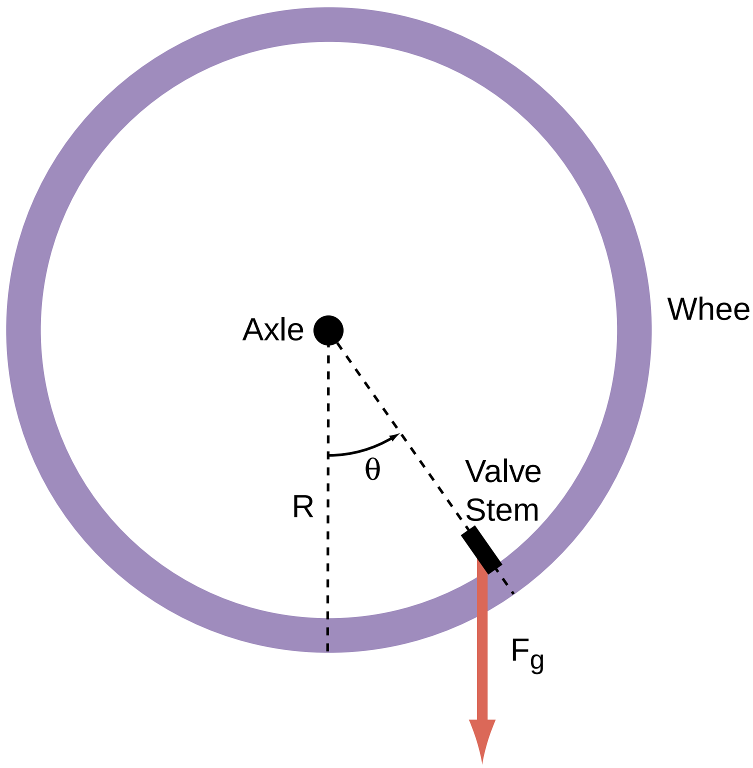

Figure 1: The bicycle wheel with a valve stem that causes a torque to act on the wheel about its axle.

enough, we can then make the approximation

$$\begin{aligned} \sin\theta &\simeq \theta & (8)\\ \Rightarrow \alpha &\simeq \frac{-Rm_v g}{I'_w}\,\theta & (9)\\ \Rightarrow \frac{d^2\theta}{dt^2} &\simeq \frac{-Rm_v g}{I'_w}\,\theta, & (10) \end{aligned}$$

which is the formula for simple harmonic motion with an angular frequency ω_w given by

$$\begin{aligned} \omega_w &= \sqrt{\frac{Rm_v g}{I'_w}} & (11)\\ &= \sqrt{\frac{(0.32)(0.025)(9.8)}{0.197}} = 0.63 \text{ rad/s}. & (12) \end{aligned}$$

This corresponds to an oscillation with a period of about 10 seconds.

13.3 More Moments of Inertia

In order to obtain more familarity with moments of inertia, let's calculate a few for relatively simple objects.

13.3.1 A Rod

Suppose we have a uniform rod of length l and mass M. Let's calculate its moment of inertia about an axis perpendicular to the rod and through its centre. Let A be the cross-sectional area of the rod. We turn to the general formula for the moment of inertia of an extended object

$$I = \int_V r^2 \rho(\vec{r})\, dV \tag{13}$$

and note that in this case the mass density ρ is constant. This means that the only thing that varies in this integral is the distance the mass in the rod is away from the centre point.

We can therefore break up the rod into very short length intervals dx, as shown in Fig. 2, and perform the integral by noting that the volume element associated with the length interval dx is just $dV = A\, dx$. The volume of the rod is $V = Al$, so the mass density is

$$\rho = \frac{M}{Al}. \tag{14}$$

Then the moment of inertia will be

$$\begin{aligned} I_r &= \int_{\text{rod}} r^2 \rho(\vec{r})\, dV && (15)\\ &= \int_{-l/2}^{+l/2} x^2 \frac{M}{Al} A\, dx && (16)\\ &= \frac{M}{l} \int_{-l/2}^{+l/2} x^2\, dx && (17)\\ &= \frac{M}{l} \left. \frac{x^3}{3} \right]_{-l/2}^{+l/2} && (18)\\ &= \frac{Ml^2}{12}. && (19) \end{aligned}$$

We see that the cross-sectional area (or the shape of the cross section) does not matter – only the length of the rod and its mass are relevant.

13.3.2 A Disk

Let's do a similar calculation, but now for a uniform disk of radius R and thickness D, with mass M. We will calculate the moment of inertia for an axis through the centre of the disk and along its axis of symmetry. In order to evaluate the volume integral that would give us the moment of inertia, we first note that the mass density is uniform and is

$$\rho = \frac{M}{\pi R^2 D}. \tag{20}$$

We also observe that all of the mass a specific distance away from the axis forms a hollow cylinder. So consider the hollow cylinder that has an inner radius r and an outer radius $r + dr$, as shown in Fig. 2. The volume of this cylinder is

$$dV = 2\pi r\, D\, dr \tag{21}$$

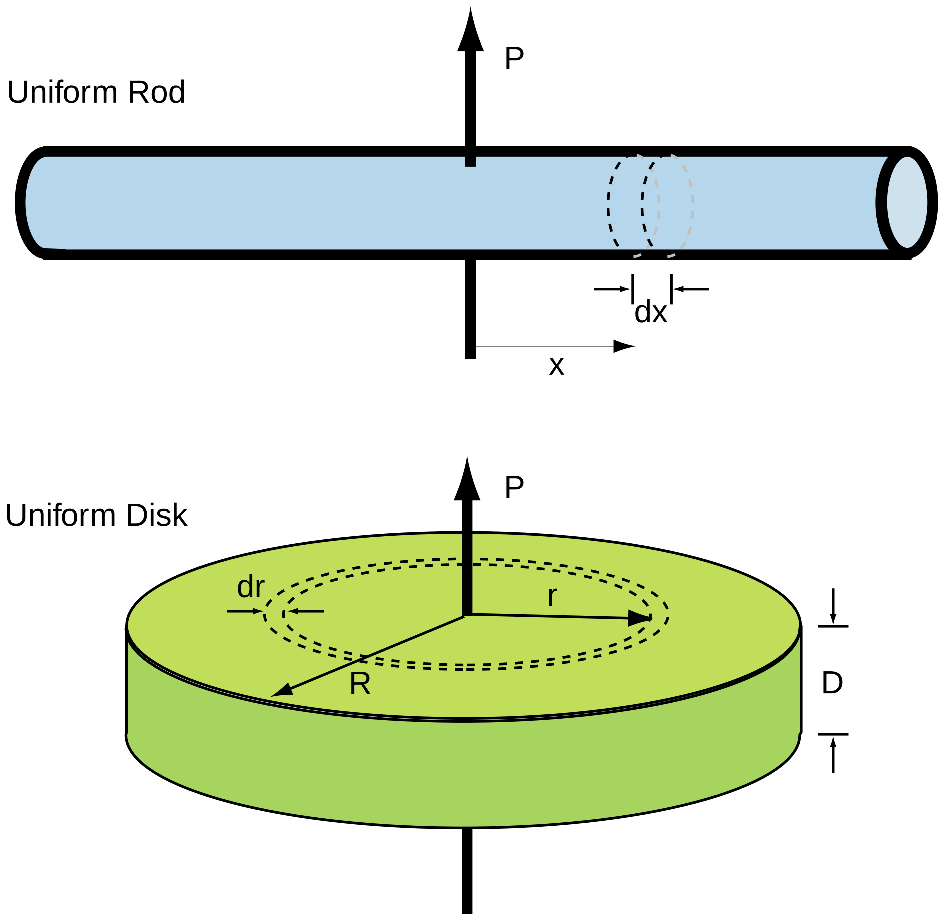

Figure 2: The calculation of the moments of inertia for a rod about an axis perpendicular to the rod passing through its centre, and for a uniform disk about an axis parallel to its axis of symmetry.

Thus the moment of inertia becomes

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\begin{aligned}
I_d &= \int_{disk} r^2 \rho(\vec{r}) dV && (22) \\
&= \int_0^R r^2 \frac{M}{\pi R^2 D} \, 2\pi D r \, dr && (23) \\
&= \frac{2M}{R^2} \int_0^R r^3 \, dr && (24) \\
&= \frac{2M}{R^2} \left. \frac{r^4}{4} \right]_0^R && (25) \\
&= \frac{MR^2}{2}. && (26)
\end{aligned}
$$

Note that this doesn't depend on the thickness of the disk D.