PHY140Y

14 Angular Motion and the Parallel Axis Theorem

14.1 Overview

- Example: Satellite spinning down
- Parallel axis theorem

14.2 Satellite Spinning Down

As a concrete example of an application of the ideas of torque and moments of inertia, let's look at the case of a cylindrical satellite spinning about its axis of symmetry, as shown in Fig. 1. The satellite has a mass $M = 940$ kg, has a diameter of 1.4 m (so its radius is $R = 0.7$ m), and is initially spinning at a rate of 10 revolutions per minute (rpm). We can assume that the satellite has a uniform mass density. It also has two thrusters located on opposite sides of the satellite. When fired, each thruster produces a force $F_j = 20$ N. We are asked how long would the thrusters have to be fired in order to stop the spinning of the satellite.

Before delving into this problem, it is useful to consider why satellites would be launched with a spin anyways. The answer is that a spinning object has associated with a dynamic stability that will tend to keep the axis about which it is spinning oriented in the same direction. We will see later that this is directly related to the conservation of angular momentum. For now, you should just note that this is in fact a useful property of rotating objects (and something that we employ when we throw circular things such as frisbees).

I'll denote the spin axis by $\hat{P}$ and assume that the initial angular velocity ω_i is along this axis (ie, the satellite is spinning counter-clockwise as viewed from the top). Let's first compute the moment of inertia, I, of the satellite. Since it is a cylinder spinning about the axis of symmetry, the moment of inertia is

$$I = \frac{MR^2}{2}. \tag{1}$$

When the two thrusters fire, the torque produced by the thrusters is

$$\begin{aligned} \vec{\tau}_j &= \vec{r}_1 \times \vec{F}_{j1} + \vec{r}_2 \times \vec{F}_{j2} && (2) \\ &= -2RF_j\hat{P}. && (3) \end{aligned}$$

The angular acceleration is therefore along the $\hat{P}$ axis and satisfies

$$\begin{aligned} \vec{\tau} &= I\vec{\alpha} && (4) \\ \Rightarrow \vec{\alpha} &= \frac{-2RF_j}{MR^2/2}\,\hat{P} && (5) \\ &= \frac{-4F_j}{MR}\,\hat{P}. && (6) \end{aligned}$$

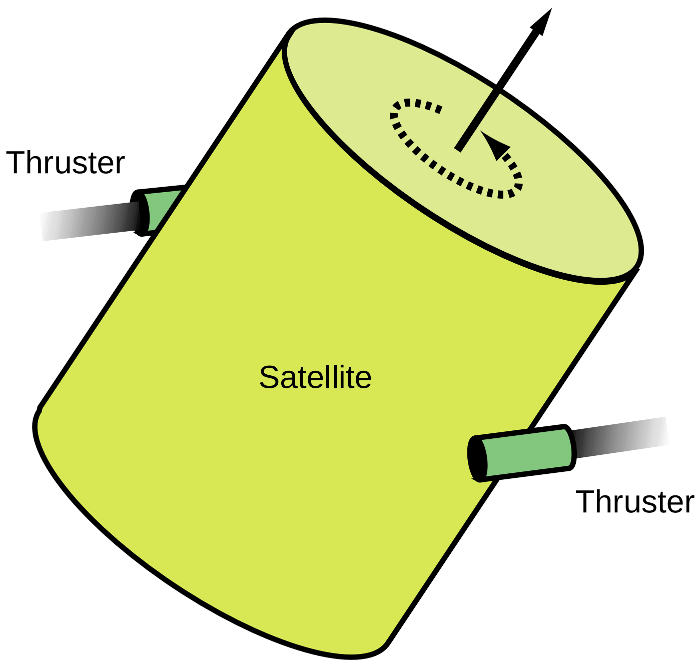

Figure 1: A satellite spinning down by firing a pair of thrusters.

We can express this is a differential equation for the angular velocity $\omega(t)$ as a function of time:

$$\frac{d\omega}{dt} = \frac{-4F_j}{MR} \tag{7}$$

$$\Rightarrow \omega(t) = \omega_i - \frac{-4F_j}{MR}\, t. \tag{8}$$

We are now only left with determining the time t_f when the angular velocity is zero. Thus,

$$\omega(t_f) = 0 = \omega_i - \frac{-4F_j}{MR}\, t_f \tag{9}$$

$$\Rightarrow t_f = \frac{\omega_i MR}{4F_j} \tag{10}$$

$$= \frac{(1.047)(940)(0.7)}{4(20)} = 8.6 \text{ s}. \tag{11}$$

14.3 Parallel Axis Theorem

Our introduction of the moment of inertia made no assumptions about where the axis of rotation was located. However, we know that the moment of inertia does depend on its position and orientation. For objects with certain symmetries, it is in fact possible to easily relate moments of inertia about different axes. The **Parallel Axis Theorem** provides one such relationship.

14.3.1 Centre of Mass

We first have to define what we mean by the centre of mass of an object. Let's look at an extended object V that is experiencing a force that acts on the various elements of V, possibly in different ways. To see this more clearly, let's divide up the object V into a number of small volume elements dV_i, with each volume element having a mass m_i and being located a position $\vec{r}_i$ from an origin O, as shown in Fig. 2. Each volume element dV_i experiences a force $\vec{F}_i$, and therefore would experience an acceleration $\vec{a}_i$.

The total force acting on this object will be

$$\vec{F}_{tot} = \sum_i m_i \vec{a}_i \tag{12}$$

$$= \sum_i m_i \frac{d^2\vec{r}_i}{dt^2} \tag{13}$$

$$= M\left[\frac{d^2}{dt^2}\sum_i \frac{m_i\vec{r}_i}{M}\right], \tag{14}$$

where we have introduced the total mass of the object, M, and have just rearranged the order of the summation and differentiation. Now define the position vector

$$\vec{R}_{cm} \equiv \sum_i \frac{m_i\vec{r}_i}{M} \tag{15}$$

as the "centre of mass" of the object V. With this definition, we see that

$$\vec{F}_{tot} = M\frac{d^2\vec{R}_{cm}}{dt^2}, \tag{16}$$

which means that the total force applied to the object serves to accelerate the point defined by $\vec{R}_{cm}$ – ie, all the forces appear to act just through that point.

This definition of the centre of mass in fact is what we normally would call the "centre of gravity" of an object. However, we see that it is defined not in the context of gravity itself, but in a more general way that doesn't depend on the nature of the applied force. However, the centre of mass has some special features. One has to do with moments of inertia calculated about axes that pass through the centre of mass.

14.3.2 Moments of Inertia Through the Centre of Mass

Let's suppose that we have an axis $\hat{P}$ that passes through the centre of mass of an object V, as shown in Fig. 3. Let I be the moment of inertia about the axis. Now suppose we have a second axis, $\hat{P}'$, that is paralle to $\hat{P}$ and located a distance $\vec{h}$ away from the centre of mass. Let's calculate the moment of inertia, I', about this new axis.

From the definition of the moment of inertia, we have

$$I' = \int_V r^2\rho(\vec{r})dV, \tag{17}$$

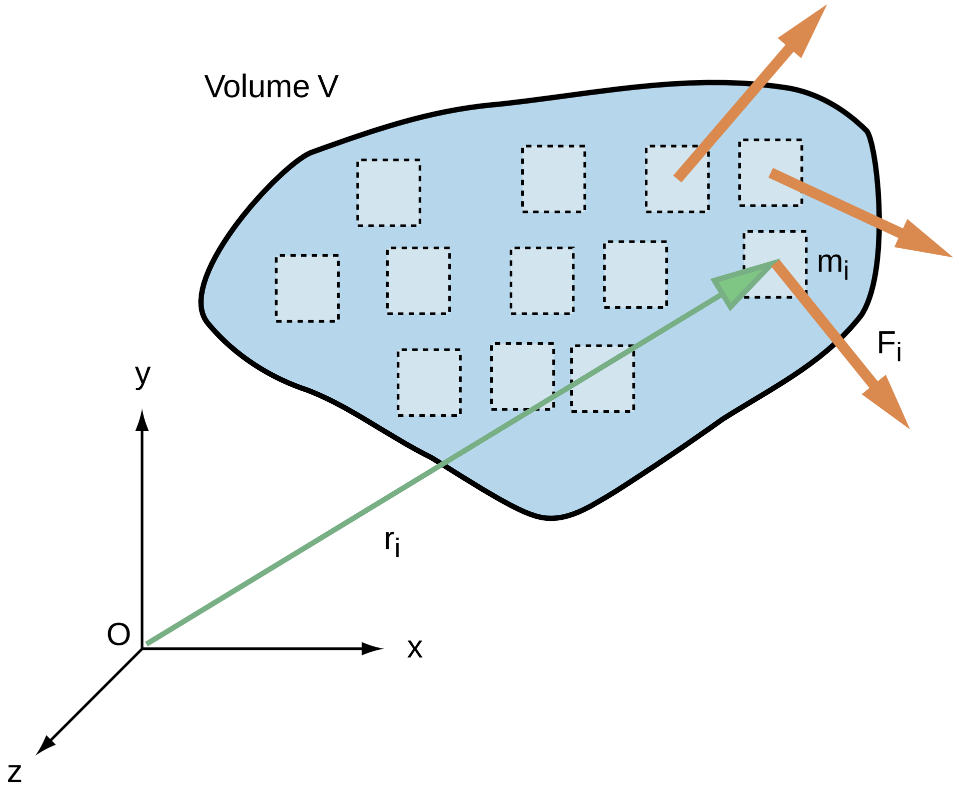

Figure 2: An arbitrary object V divided up into a set of smaller volume elements, dV_i, each located at a position $\vec{r_i}$ from an origin O.

where $\vec{r}$ is the distance from the axis $\hat{P}'$ as shown in Fig. 3. We can rewrite this vector $\vec{r}$ as the position of the same point defined relative to the centre of mass, $\vec{r}_{cm}$, so that

$$\begin{aligned} I' &= \int_V |\vec{r}_{cm} - \vec{h}|^2 \rho(\vec{r}_{cm}) dV && (18) \\ &= \int_V \left[r_{cm}^2 + h^2 - 2\left(\vec{r}_{cm} \cdot \vec{h}\right)\right] \rho(\vec{r}_{cm}) dV && (19) \\ &= \int_V r_{cm}^2 \rho(\vec{r}_{cm}) dV + \int_V h^2 \rho(\vec{r}_{cm}) dV - 2\int_V \left(\vec{r}_{cm} \cdot \vec{h}\right) \rho(\vec{r}_{cm}) dV. && (20) \end{aligned}$$

The first integral is just the moment of inertia about the axis $\hat{P}$. The second term is in fact just $h^2 M$, since $\vec{h}$ is a constant vector. The last term is in fact

$$\begin{aligned} 2\int_V \left(\vec{r}_{cm} \cdot \vec{h}\right) \rho(\vec{r}_{cm}) dV &= 2\vec{h} \cdot \left(\int_V \vec{r}_{cm} \rho(\vec{r}_{cm}) dV\right) && (21) \\ &= 2\vec{h} \cdot 0 = 0, && (22) \end{aligned}$$

since the integral on the right-hand side is in fact just the definition of the centre of mass relative to itself, which is a zero vector!

Thus, the moment of inertia about the parallel axis $\hat{P}'$ is

$$I' = I + h^2 M, \tag{23}$$

a simple and useful result.

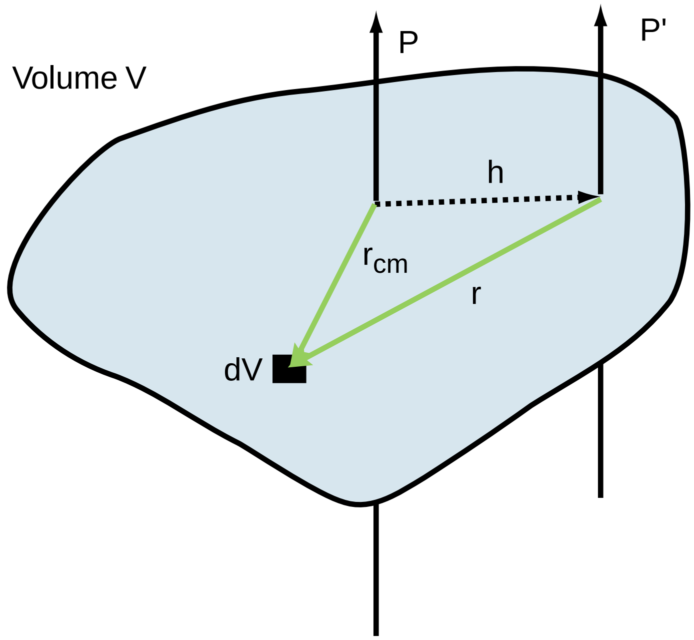

Figure 3: The calculation of the moment of inertia about an axis $\hat{P}'$ parallel to the axis $\hat{P}$ that passes through the centre of mass.

14.3.3 Moment of Inertia of Rod

As a concrete example, let's revisit the calculation of the moment of inertia of the rod of length l about the axis through its centre, $\hat{P}$. To determine the moment of inertia of the rod about an axis at the end of the rod and parallel to $\hat{P}$, we just note that this new axis is a distance $h = l/2$ from the centre of mass. Thus, the new moment of inertia is

$$I' = I + \left(\frac{l}{2}\right)^2 M \tag{24}$$

$$= \frac{Ml^2}{12} + \frac{Ml^2}{4} \tag{25}$$

$$= \frac{Ml^2}{3}. \tag{26}$$