PHY140Y

15 Work, Energy and Rolling Motion

15.1 Overview

- Rotational kinetic energy
- Work performed under a torque
- Rolling motion

15.2 Rotational Energy

It's now time to figure out how to relate the work done on a rotating object, with the kinetic energy of its motion, and with the torques that are placed on it. To do this, let's consider our typical rigid body V rotating about an axis $\hat{P}$, as shown in Fig. 1. Suppose it has instantaneous angular velocity ω. Let's break up the body into a large number of small pieces, dV_i, each with mass ρdV_i, where ρ is the mass density (which for simplicity we will assume is constant). The instantaneous linear velocity of a given mass ρdV_i, located $\vec{r}_i$ from the $\hat{P}$, will then be

$$v_i = |\vec{r}_i|\omega \tag{1}$$

so the kinetic energy of this mass will be

$$\begin{aligned} dK_i &= \frac{1}{2}(\rho dV_i)v_i^2 && (2)\\ &= \frac{1}{2}(\rho dV_i)|\vec{r}_i|^2\omega^2 && (3)\\ &= \frac{1}{2}\omega^2|\vec{r}_i|^2\rho\, dV_i. && (4) \end{aligned}$$

The total energy of rotation is obtained by just summing all the individual volume elements, or in the limit of very small volume elements, performing the volume integral

$$\begin{aligned} K &= \int_V \frac{1}{2}\omega^2|\vec{r}_i|^2\rho\, dV_i && (5)\\ &= \frac{1}{2}\omega^2\int_V |\vec{r}_i|^2\rho\, dV_i && (6)\\ &= \frac{1}{2}I\,\omega^2, && (7) \end{aligned}$$

where we have noticed that the volume integral is just the moment of inertia of the object about the axis $\hat{P}$.

To get some feeling for how large this can be, let's consider the rotational kinetic energy of the student who we had spinning on a chair a few lectures ago. If we assume the student had a mass

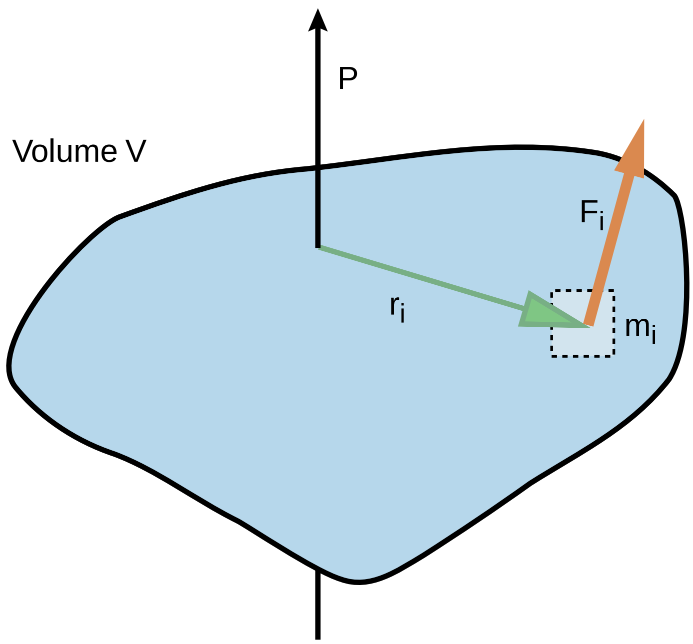

Figure 1: Calculating the energy of a rotating rigid body.

$M = 70$ kg and could be reasonably approximated by a uniform cylinder of radius $R = 30$ cm, then the moment of inertia of the student spinning about his axis would be that of a uniform disk, ie.

$$I_s = \frac{MR^2}{2} \tag{8}$$

$$= \frac{(70)(0.3)^2}{2} = 3.2 \text{ kg m}^2. \tag{9}$$

I estimated that the student was spinning at a rate of about 3 rad/s (or about 1 revolution every 2 seconds), so that the rotational energy of the student would be

$$K_s = \frac{1}{2} I_s \omega_s^2 \tag{10}$$

$$= \frac{(3.2)(3)^2}{2} = 14 \text{ J}. \tag{11}$$

This is not a lot of energy, especially if you compare it with, for example, the kinetic energy of the same student walking (do it!).

15.3 Rotational Work

Recall that our definition of "work" when we looked at linear motion involved the product of the force applied over a distance. Let us do a similar calculation for a rotating object, again first breaking up the object and looking at the work performed on a volume element dV_i (again refer to Fig. 1).

If a force $\vec{F}_i$ is applied to the volume element dV_i, located $\vec{r}_i$ from the rotation axis $\hat{P}$, and the volume element moves a small distance $\Delta x = r_i d\theta$, then the work done in that motion is

$$
\begin{aligned}
dW_i &= \vec{F}_i \cdot \Delta\vec{x} && (12)\\
&= F_i r_i \, d\theta && (13)\\
&= \tau_i \, d\theta. && (14)
\end{aligned}
$$

The total work done on the entire object is then just the sum of all the torques acting on the whole object, or just

$$\Delta W = \tau \, d\theta, \tag{15}$$

and the work done in rotating the object from an angle θ_i to an angle θ_f is

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta. \tag{16}$$

Note that τ in general will depend on the displacement angle θ This is already a useful result.

To put this into a form where we can relate it to the change in kinetic energy, let's do some differential gymnastics:

$$
\begin{aligned}
\tau &= I \frac{d\omega}{dt} && (17)\\
&= I \left(\frac{d\omega}{d\theta}\right)\left(\frac{d\theta}{dt}\right) && (18)\\
&= I\omega \left(\frac{d\omega}{d\theta}\right). && (19)
\end{aligned}
$$

With this relationship, we can rewrite the work performed as

$$
\begin{aligned}
W &= \int_{\theta_i}^{\theta_f} \tau \, d\theta && (20)\\
&= \int_{\theta_i}^{\theta_f} I\omega \left(\frac{d\omega}{d\theta}\right) d\theta && (21)\\
&= \int_{\omega_i}^{\omega_f} I\omega \, d\omega && (22)\\
&= \frac{1}{2} \, I\omega^2 \Big]_{\omega_i}^{\omega_f} && (23)\\
&= \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2, && (24)
\end{aligned}
$$

which is just the change in the overall kinetic energy of the object (which is just as it should be).

So, using the original relationship between work and torque, let's see how much torque I placed on the student to get him to spin up to an angular velocity of $\omega_f = 3$ rad/s. Assuming that I

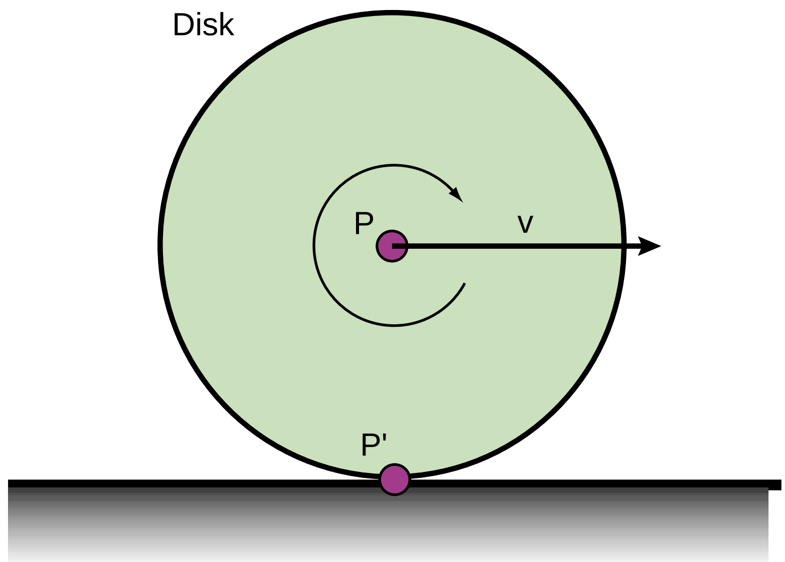

Figure 2: A disk rolling without slipping on a smooth surface.

applied a constant torque τ_P, over an angular interval $\Delta\theta = \pi/4$, then the torque applied is given by the relationship

$$\begin{aligned} W &\simeq \tau_P \Delta\theta \quad (25) \\ \Rightarrow \tau_P &= \frac{W}{\Delta\theta} \quad (26) \\ &= \frac{14}{\pi/4} = 18 \text{ N m.} \quad (27) \end{aligned}$$

15.4 Rolling Motion

We now have most of the tools that will allow us to analyze the complete motion of a rotating *and* translating object. The simplest such object is a cylinder rolling on a smooth, flat surface without slipping, as shown in Fig. 2.

Let M be the mass of the cylinder and R be its radius. If it is rotating at an angular velocity ω, then its centre of mass is moving forward at a velocity $v = R\omega$. This means that its linear kinetic energy is

$$\begin{aligned} K_{trans} &= \frac{1}{2} M v^2 \quad (28) \\ &= \frac{MR^2}{2} \omega^2. \quad (29) \end{aligned}$$

The moment of inertia of the disk about its axis of symmetry is

$$I_d = \frac{MR^2}{2}, \quad (30)$$

so the kinetic energy of rotation is

$$\begin{aligned} K_{rot} &= \frac{1}{2}I\omega^2 && (31)\\ &= \frac{MR^2}{4}\omega^2. && (32) \end{aligned}$$

The total kinetic energy of motion is

$$\begin{aligned} K_{tot} &= K_{trans} + K_{rot} && (33)\\ &= \frac{MR^2}{2}\omega^2 + \frac{MR^2}{4}\omega^2 && (34)\\ &= \frac{3MR^2}{4}\omega^2, && (35) \end{aligned}$$

which is exactly 50% larger than the kinetic energy of the cylinder if it was just sliding forward on a frictionless surface.

We can take a rather different view of this altogether. Consider the point of instantaneous contact of the cylinder on the smooth surface. Instead of looking at the motion of the cylinder as rotational motion about the axis of the cylinder in association with the translational motion of its centre of mass, it is *equivalent* to think about the motion as being just a rotation about the axis parallel to the axis of symmetry of the cylinder and through the point of instantaneous contact of the cylinder with the surface. With this perspective, we are then taking into account both the rotational motion and the translational motion of the centre of mass.

To see this work out, we have to first calculate the moment of inertia of the cylinder about the axis $\hat{P}'$, which is parallel to the symmetry axis of the cylinder and through the point of instantaneous contact with the surface. The moment of inertia of the cylinder about this axis, I', is given by the parallel axis theorem:

$$\begin{aligned} I' &= I + MR^2 && (36)\\ &= \frac{MR^2}{2} + MR^2 && (37)\\ &= \frac{3MR^2}{2}. && (38) \end{aligned}$$

The angular velocity remains the same as before, so the total kinetic energy is now the rotational energy given by

$$\begin{aligned} K_{tot} &= \frac{1}{2}I'\omega^2 && (39)\\ &= \frac{3MR^2}{4}\omega^2, && (40) \end{aligned}$$

which is exactly the same result we obtained looking at the translational and rotational motions separately.