PHY140Y

16 Angular Momentum

16.1 Overview

- Conservation of angular momentum
- Example: Hydrogen atom
- Example: Two cars

16.2 Angular Momentum

We have looked at a specific kind of rotational motion, namely that associated with rotation about a specific axis. In this context, all torques, angular accelerations and angular velocities have been parallel or antiparallel to the rotation axis. We are missing one crucial concept, the rotational analogue of linear momentum.

Suppose we have a rotating object experiencing a torque $\vec{\tau}$. Let's do our usual trick and break up the object into small volume elements, each located at a distance $\vec{r}_i$ from the rotation axis, and each experiencing a force $\vec{F}_i$. Then we can write the total torque on the object as

$$\begin{aligned} \tau &= \sum_i \vec{r}_i \times \vec{F}_i && (1) \\ &= \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt}, && (2) \end{aligned}$$

where $\vec{p}_i$ is the linear instantaneous momentum of the volume element.

The cross product involves a time-derivative on the right-hand side, and a quantity ($\vec{r}_i$) on the left-hand side that also depends on time. To see how to simplify this, note that the time derivative of a cross-product of two vectors $\vec{x}$ and $\vec{y}$ is given by

$$\frac{d}{dt}\left(\vec{x} \times \vec{y}\right) = \vec{x} \times \frac{d\vec{y}}{dt} + \frac{d\vec{x}}{dt} \times \vec{y}. \quad (3)$$

(Remember to keep the order of the vectors correct, as changing their order introduces a sign change.) So, let's look at the term

$$\begin{aligned} \frac{d\vec{r}_i}{dt} \times \vec{p}_i &= \vec{v}_i \times \vec{p}_i && (4) \\ &= \frac{\vec{p}_i}{m_i} \times \vec{p}_i = 0, && (5) \end{aligned}$$

since the cross-product of two parallel vectors is 0.

This means that we can rewrite Eq. 2 as

$$\vec{\tau} = \sum_i \left(\vec{r}_i \times \frac{d\vec{p}_i}{dt} + \frac{d\vec{r}_i}{dt} \times \vec{p}_i \right) \tag{6}$$

$$= \sum_i \frac{d}{dt} (\vec{r}_i \times \vec{p}_i) \tag{7}$$

$$= \frac{d}{dt} \sum_i (\vec{r}_i \times \vec{p}_i) . \tag{8}$$

With this result, it is natural to define the **angular momentum**

$$\vec{L} \equiv \sum_i (\vec{r}_i \times \vec{p}_i) , \tag{9}$$

which has units of $\mathrm{kg\,m^2/s}$. With this defintion, we can write for the rotational motion of a rigid body

$$\vec{\tau} = \frac{d\vec{L}}{dt}. \tag{10}$$

This has significance consequences.

1. If the axis of rotation is not changing, then

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{11}$$

$$= I\frac{d\vec{\alpha}}{dt} \tag{12}$$

$$\Rightarrow \vec{L} = I\vec{\omega}, \tag{13}$$

   which gives us a very quick way of calculating the angular momentum of an object.

2. If there are no torques acting on the object, then the angular momentum is conserved and is a constant of motion. This is analogous to the conservation of linear momentum that we studied many moons ago.

16.3 Example: Hydrogen Atom

We will show in a few lectures that one of the lowest energy excited states of the hydrogen atom is one where the angular momentum of the electron orbiting the proton is

$$|\vec{L_e}| = \sqrt{2}\hbar = \sqrt{2}(1.06 \times 10^{-34})\ \mathrm{kg\,m^2/s}, \tag{14}$$

where $\hbar$ is what is known as Planck's constant and has units of angular momentum. The classical radius of orbit of the electron is the Bohr radius

$$a_0 = 5.29 \times 10^{-11}\ \mathrm{m}. \tag{15}$$

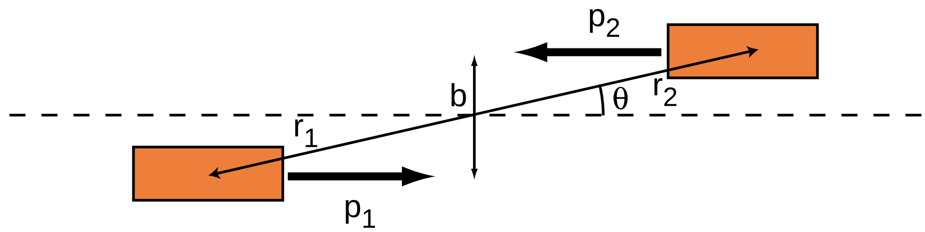

Figure 1: A “rotating” system of two cars passing each other.

If the momentum of the electron is $p_e = m_e v$ and the orbit is circular, the magnitude of the angular momentum of the electron is

$$
\begin{aligned}
L_e &= |\vec{r} \times \vec{p}_e| && (16)\\
&= a_o m_e v && (17)\\
\Rightarrow v &= \frac{L_e}{a_0 m_e} && (18)\\
&= \frac{\sqrt{2}(1.06 \times 10^{-34})}{(5.29 \times 10^{-11})(9.11 \times 10^{-31})} = 3.1 \times 10^6 \text{ m/s.} && (19)
\end{aligned}
$$

This is about 1% of the speed of light! Although this is a purely classical result, it gives us some idea of the dynamics of the electron orbiting a proton.

16.4 Example: Two Cars

As another example, let's consider the case of two cars approaching each other on a two-lane highway, as shown in Fig. 1. Each car has a mass $m_c = 1500$ kg and a speed $v_c = 100$ km/h. The centres of mass of the two cars are separated by a distance $b = 5$ m. Let's see what angular momentum tells us about this system.

Let's first suppose the cars are located as shown in Fig. 1. Then the angular momentum of the two car system about the axis mid-way between each car is

$$
\begin{aligned}
\vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i && (20)\\
&= r_1 \sin(\pi - \theta)\, p_1 + r_2 \sin(\pi - \theta)\, p_2 && (21)\\
&= r_1 \sin\theta p_1 + r_2 \sin\theta p_2 && (22)\\
&= \frac{b}{2} p_1 + \frac{b}{2} p_2 && (23)\\
&= bp = 2.1 \times 10^5 \text{ kg m}^2\text{/s.} && (24)
\end{aligned}
$$

We see that the angular momentum is a constant, which in fact we could have deduced from the observation that there were no forces (and therefore no torques) acting on the two-car system.

Now think about what happens if the lateral separation of the cars is such that their corners collide as they pass each other and the two vehicles “fuse” together and begin rotating. As a homework exercise, estimate how fast the two cars would rotate. You may have to make some approximation for the moment of inertia of the two-car system.