PHY140Y

2 Measuring G and Gravitational Potential Energy

2.1 Overview

- Measurement of G
- Gravitational Potential Energy
- The Truth About Escape Velocities

2.2 Measuring G

When Newton put forward his law of universal gravitation aroung 1670, he was not able to give a precise value for the constant of universal gravitation G. This had to wait over a hundred years till 1798, when Lord Henry Cavendish made the first accurate determination of G.

Why is it so difficult to measure G? Think again about the nature of gravity. We are aware of it because of our own experience of its force on the surface of the Earth and we can accurately measure the acceleration of gravity. But from the law of universal gravitation, what we know is

$$g = \frac{GM_E}{R_E^2}. \tag{1}$$

We see that if we know the radius of the Earth, R_E, we can only infer GM_E, namely the G times the mass of the Earth. Similarily, knowing the moon's orbit, it's radius and an estimate for it's mass, we only infer GM_E. A direct measurement of the force of gravity, using two masses of know size is required. And because gravity is so weak, designing a sufficiently sensitive instrument is a challenge.

Cavendish addressed this by developing what we call the "Cavendish Balance," shown in Fig. 1. He fastened two 5-cm lead balls together with a rod, and then fastened the rod to a thin fibre that he used to suspend them. He then brought two large spheres in proximity to each of the two lead spheres on the "dumbell." The force of gravity caused the masses to attract one another, twisting the fibre. By knowing the torsion coefficient on the fibre, he was able to estimate a value for the force. This allowed him to directly calculate G.

The most precise estimate of G is

$$G = (6.67259 \pm 0.00085) \times 10^{-11}\ \mathrm{N\ m^2/kg^2}. \tag{2}$$

The precision to which this is measured, about 130 parts in a million, is not outstanding. It turns out to be a difficult constant to measure because it is so small. Note the units of G!

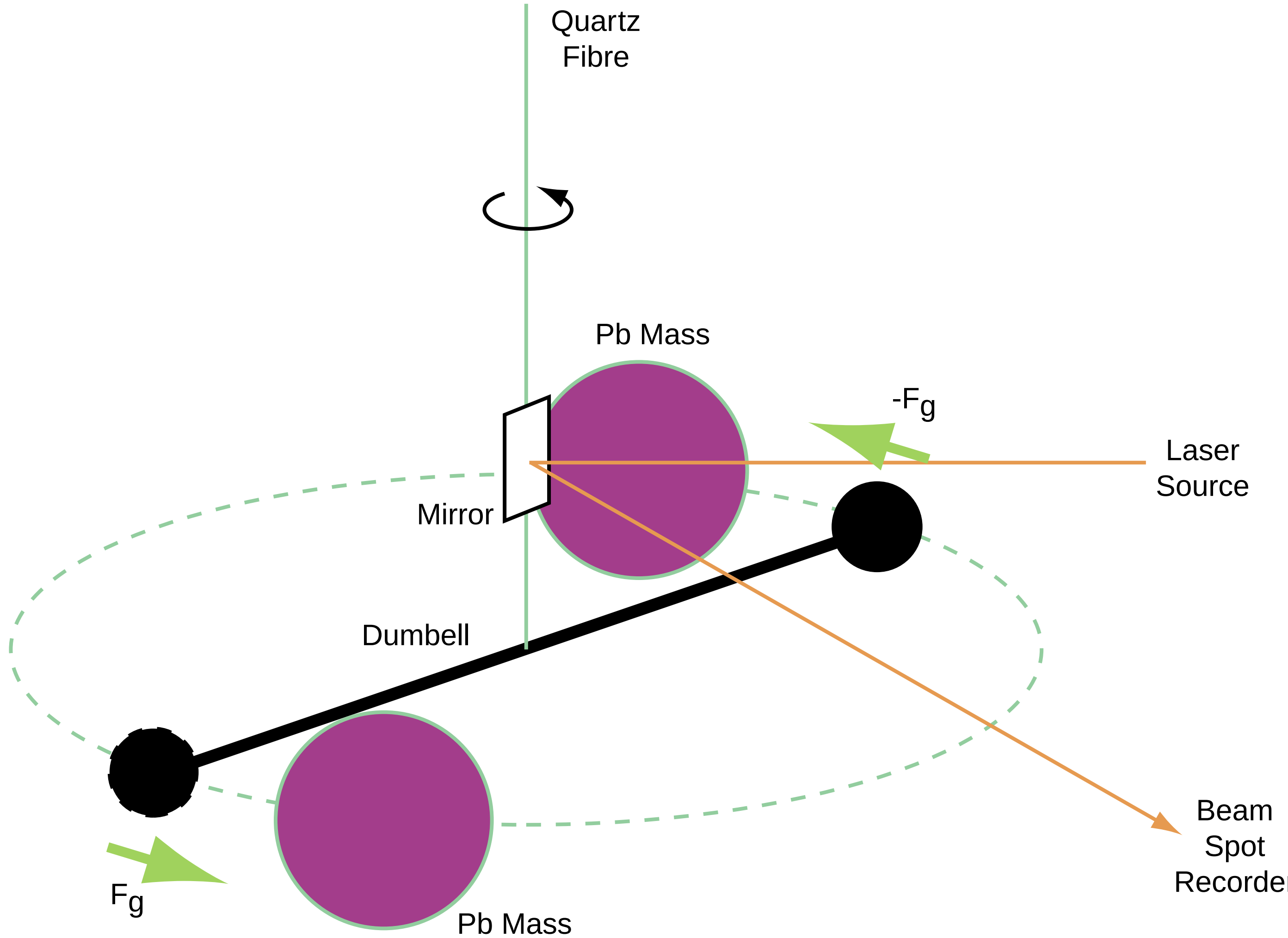

Figure 1: A schematic of the Cavendish Balance, as seen in our first and second-year laboratory.

2.3 Gravitatational Potential Energy

Once we have defined a force of gravity, the next natural step from a dynamics point of view is to introduce the concept of energy into the picture. We have already seen that in the context of a *constant* acceleration due to gravity, g, (which is only true over distances small compared to the radius of the object doing the attracting), we can define a gravitational potential energy

$$U \equiv mgh, \tag{3}$$

where m is the mass of the object and h is the distance above some reference point where we define $U \equiv 0$.

To generalize this, we start with a massive object with mass M, and find a way of calculating the work done to move an object in its gravitational field from one place to another. It is clear that if we move an object and keep it at the same proximity or radius from the large mass, we do not gravitational work as whatever motion has to take place happens at right angles to the force of gravity. Thus, we would only do work with we move the object in radius. So, suppose we have to move the object from point $\vec{r_1}$ to point $\vec{r_2}$ in the gravitational field. We then find that the work we have to do is given by

$$\begin{aligned} W &= \int_{r_1}^{r_2} \mathbf{F} \cdot \mathbf{dr} && (4) \\ &= -\int_{r_1}^{r_2} \frac{GMm}{r^2}\, dr && (5) \end{aligned}$$

$$= \left. \frac{GMm}{r} \right]_{r_1}^{r_2} \tag{6}$$

$$= GMm \left[\frac{1}{r_2} - \frac{1}{r_1} \right]. \tag{7}$$

We see that the work we have to do only depends on the initial and final radii of the mass m, which tells us that it doesn't matter which we path we chose to move the mass.

So the work we do to move the object goes into increasing the object's gravitational potential energy, U. We therefore can relate W to the negative of the change in the object's potential energy:

$$U(r_2) - U(r_1) = -GMm \left[\frac{1}{r_2} - \frac{1}{r_1} \right] \tag{8}$$

$$U(r_2) = -\frac{GMm}{r_2} + \text{constant}, \tag{9}$$

where the constant in the right hand side is determined by our choice of r_1. This choice is in fact totally arbitrary, since in principle, it is only the change in potential energy that has any physical significance, because it is that change that we relate to the work performed on the object (and therefore the forces acting on it).[1] It is simplest and most intuitive to chose $r_1 = \infty$ as that means the constant is zero and we can ignore it.[2]

This is now the generalization of our original gravitational potential energy. We now see that

$$g = \frac{GM_E}{R_E}, \tag{10}$$

where $M_E = 5.97 \times 10^{24}$ kg is the mass of the Earth and $R_E = 6.37 \times 10^6$ m is the radius of the Earth (check it out!).

2.4 Escape Velocity

Let's suppose I am sitting out in space at an initial radius r_i far away from the Earth (let's assume $r_i >> R_E$). If I start from at rest, the I will begin to accelerate toward the Earth under the force of gravity. If we assume that air resistance is neglible, then I will end up impacting the surface of the Earth with a considerable velocity. Let's work out the details.

My total energy when we start this is

$$E_i = K + U \tag{11}$$

$$= 0 - \frac{GM_E m_p}{r_i}, \tag{12}$$

where m_p is my mass (which I claim shouldn't matter). When I impact the Earth, my total energy will still be the same (remember, we are ignoring important effects such as air resistance). Thus,

[1] Can you think of a counter-example to this? See if there is any simple system where the absolute value of the potential energy makes any difference to what you would observe.

[2] This is the only general choice that makes any sense to me, since $r_1 = 0$ is not a possibility and any other value of r_1 would be totally arbitrary.

my velocity on impact, v_f, will be given by conservation of total energy,

$$\begin{aligned}
E_i = E_f &= \frac{1}{2}m_p v_f^2 - \frac{GM_e m_p}{R_E} && (13)\\
\Rightarrow \frac{1}{2}m_p v_f^2 &= GM_e m_p \left(\frac{1}{R_E} - \frac{1}{r_i}\right) && (14)\\
\Rightarrow v_f &= \sqrt{\frac{2GM}{R_E}\left(1 - \frac{R_E}{r_i}\right)}. && (15)
\end{aligned}$$

Since the distance I start from, r_i, is so much larger than R_E, I can ignore the term in the parentheses and arrive at an expression for my impact velocity

$$v_f = \sqrt{\frac{2GM}{R_E}} = 1.12 \times 10^4 \text{ m/s} = 11.2 \text{ km/s}. \qquad (16)$$

This velocity is usually referred to as the “escape velocity,” v_{esc}, because it would be the velocity at which I would have to leave the surface of the Earth in order to “escape” from the Earth’s gravitational field. In this sense, it is a curious number as it is not really relevant when we want to lauch a rocket.

You can see this by asking how long, under a reasonable acceleration like $a = 10$ m/s^2, would it take for something to reach this escape velocity. It would take a time

$$\Delta t = \frac{v_{esc}}{a} \sim 1,120 \text{ s}, \qquad (17)$$

and during that time it would travel a distance

$$\Delta d = \frac{1}{2}a\,(\Delta t)^2 \sim 6.3 \times 10^6 \text{ m}. \qquad (18)$$

No rocket ever behaves in this way! What we normally do is first accelerate a rocket with a somewhat larger acceleration (3-4 g’s is about typical) for a period of several minutes to get it into a low Earth orbit of order 500 km in one stage and then take it out of Earth orbit with a second stage that allows it to achieve the necessary velocity to take it out of Earth’s gravitational field. It is at this stage where much higher velocity’s are attained, in part because one can throw away the first stage of the rocket that got the vehicle out of the Earth’s atmosphere.