PHY140Y

24 QM Oscillator

24.1 Overview

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24.2 QM Oscillator

The particle in a one-dimensional box is perhaps one of the simplest systems that yields a non-trivial quantum mechanical solution. Another simple system is an oscillator, like a mass on a spring, as shown in Fig. 1.

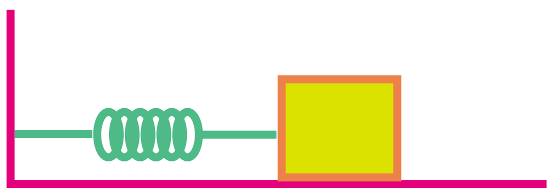


Figure 1: The typical mass on a spring displaced from it's equilibrium position by a distance x.

The classical equation of motion for this system comes directly from Newton's Second Law:

$$F = ma \tag{1}$$

$$\Rightarrow -kx = m\frac{d^2x}{dt^2} \tag{2}$$

$$\Rightarrow \frac{d^x}{dt^2} = -\left(\frac{k}{m}\right)x. \tag{3}$$

The solutions for this are the periodic sine and cosine functions

$$x(t) = A\sin(\omega t) + B\cos(\omega t), \tag{4}$$

and by substitution, we find that

$$\omega^2 = \frac{k}{m} \tag{5}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} \tag{6}$$

is the natural angular frequency for the oscillation.

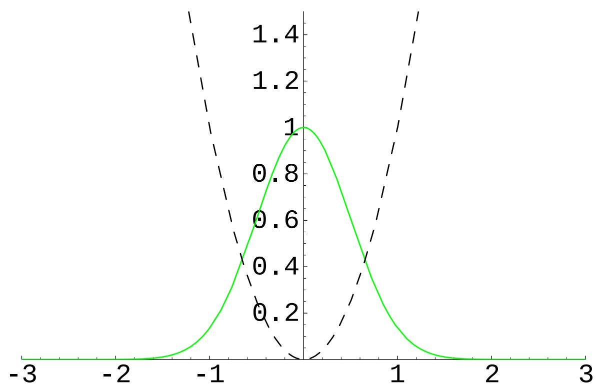


Figure 2: The square of the wave function (solid) for the ground state plotted along-side the potential energy function (dashed). The units of x are $\sqrt{\hbar/m\omega}$, and in these units $|x| = 1$ is the largest value a classical oscillator would have if you believe in Conservation of Energy.

To quantize this problem, we note that the potential energy for this system is

$$U(x) = \frac{1}{2}kx^2. \tag{7}$$

This means that the Schrödinger equation for this system is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x). \tag{8}$$

What can we say about the boundary conditions on the wave function $\psi(x)$? Well, we know that we expect the probability of finding the particle to vanish as $|x| \to \infty$ and in fact that is about all we need to know to solve for $\psi(x)$.

24.3 Characteristics of Solutions

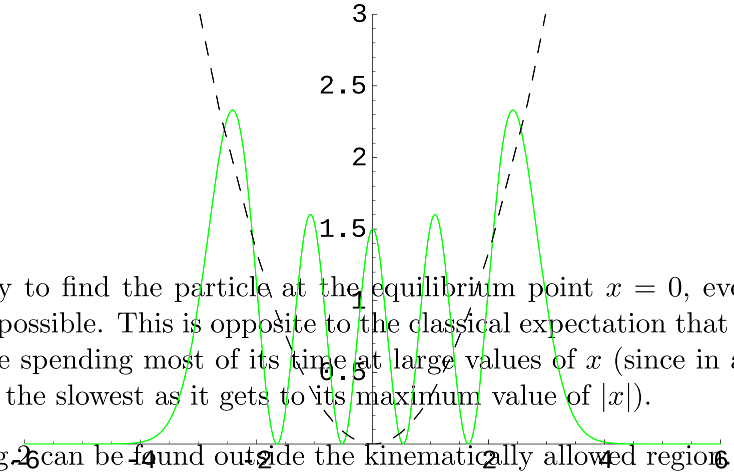
The actual techniques by which we would solve a differential equation like Eq. 8 are somewhat advanced, so we will not get into them here. We shall note that the form for the lowest energy solution to the Schrödinger equation is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/(2\hbar)} \tag{9}$$

and the energy of this state is $E_0 = \hbar\omega/2$. The square of this ground state wave function is plotted in Fig. 2, along side the potential energy function.

This solution already has some interesting features:

1. The ground state energy is not zero, but $\hbar\omega/2$. Thus, quantum mechanics predicts that the oscillator will never be in a state of total rest.



2. We are most likely to find the particle at the equilibrium point $x = 0$, even though it has the lowest energy possible. This is opposite to the classical expectation that we would expect to find the particle spending most of its time at large values of x (since in an oscillator, the particle is moving the slowest as it gets to its maximum value of $|x|$).

3. The particle in Fig. 2 can be found outside the kinematically allowed region. In this plot, the maximum classical displacement is $x = 1$. The fact that some fraction of the time we would expect to see the particle violate energy conservation is an important observation.

The energy of the higher level states of this system are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{10}$$

The probability density for the higher energy states begins to show more of the expected classical behaviour. For example, I illustrate the $n = 4$ state in Fig. 3: We now see a greater likelihood of finding the particle near the extreme range of motion, as would be the classical result. We also see that the particle still may be found outside the classical allowed region, a phenomenon known as "tunnelling."

Figure 3: The square of the wave function (solid) for the $n = 4$ state plotted along-side the potential energy function (dashed). The units of x are in $\sqrt{\hbar/m\omega}$. In these units, the classical maximum value of $|x|$ is 3, based on conservation of energy.