PHY140Y

28 Angular Momentum in the Hydrogen Atom

28.1 Overview

- Angular Dependence in the Schrödinger's Equation
- Angular Momentum Substates

28.2 Angular Dependence in the Schrödinger Equation

Our solutions for the hydrogen atom wave functions have, up till now, assumed that the wave function has no dependence on θ or ϕ. The fact that we have found an infinite set of wave functions that satisfy this rather restrictive criteria is perhaps unexpected. Let's now consider what happens when we relax this condition.

The mathematics of how to solve for the angular dependence takes us well beyond this course, so I will only sketch out the basic approach. We make a "guess" that a solution to the Schrödinger equation has the form

$$\psi(r,\theta,\phi) \quad = \quad R(r)\Theta(\theta)\Phi(\phi), \tag{1}$$

where $R(r)$ is just a function of the variable r, where $\Theta(\theta)$ is just a function of the variable θ and where $\Phi(\phi)$ is just a function of the variable ϕ. When we substitute this ansatz into Schrödinger's equation, we obtain separate differential equations for $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$ that have analytic solutions. This procedure is known as "Separation of Variables."

Let's now turn to the properties of these solutions.

28.3 Angular Momentum Substates

First, I note that the solutions for $R(r)$ are the same wave functions that we had obtained assuming no angular dependence, which we could specify using the principal quantum number n. The second observation is that the energy of the quantum state only depends on n and is the same energy level we calculated before.

Second, for each value of n, there are a set of new solutions that we can characterize by two new quantum numbers, l and m_l:

- l, known as the orbital quantum number, can take on values $l = 0, 1, \ldots, n-1$.
- m_l, known as the orbital magnetic quantum number, can take on values $-l, -l+1, \ldots, l-1, l$, ie., it can take on $2l+1$ possible values.

As their names suggest, l and m_l specify states of the hydrogen atom that have definite orbital angular momentum, which we can denote by the orbital angular momentum vector $\vec{L}$. We find that $\vec{L}$ is quantized in a very peculiar way:

- The magnitude of the orbital angular momentum is given by

$$|\vec{L}| \quad = \quad \sqrt{l(l+1)}\hbar, \tag{2}$$

  where $\hbar$ is Planck's constant. It is remarkable that it reappears here in this context, given that it was originally introduced to quantize the energy of atomic states. Traditionally, we denote states with specific values of l by a letter: $l = 0$ is denoted "S," $l = 1$ is denoted "P," $l = 2$ is denoted "D," and $l = 3$ is denoted "F". The higher l states then follow with designations "H, I, J, ..."

- The quantum number m_l specifies the magnitude of the projection of $\vec{L}$ on the coordinate axis $\hat{z}$:

$$L_z \quad = \quad m_l\hbar. \tag{3}$$

  Thus it tells us that the magnitude of this projection can only take on $2l + 1$ possible values.

Why is the axis $\hat{z}$ singled out in this problem? In fact, it is up to us to choose a $\hat{z}$ first, and once having made that choice we can identify the specific angular momentum states relative to that axis. What quantum mechanics is telling us is that the most one can learn about the angular momentum is by allowing us to specify those wave functions that have the property that L_z is quantized. What it also tells us is that this is as much as we can learn about the angular momentum, namely its total magnitude and one of its components. This means that there are two other components we cannot specify because they are not quantized. A rather peculiar situation, but one that has been verified by a number of experiments starting in the early 1930's.

Thus, the possible wave functions of the hydrogen atom are now given by specifying the quantum numbers n, l and m_l. So for example, the wave functions with $n \leq 3$ are given by

$$n = 1 \quad \text{and} \quad l = 0, m_l = 0 \tag{4}$$
$$n = 2 \quad \text{and} \quad l = 0, m_l = 0 \tag{5}$$
$$l = 1, m_l = -1, 0, \text{ and } 1 \tag{6}$$
$$n = 3 \quad \text{and} \quad l = 0, m_l = 0 \tag{7}$$
$$l = 1, m_l = -1, 0, \text{ and } 1 \tag{8}$$
$$l = 2, m_l = -2, -1, 0, 1 \text{ and } 2. \tag{9}$$

The picture we have of the angular momentum vector $\vec{L}$ is summarized in Fig. 1, where we display the possible orientations for the 3 angular momentum substates with $l = 1$. It is important to recognize that although we know exactly what the values of L_z and $|\vec{L}|$ are because they are quantized, we cannot say anything about L_x or L_y individually.

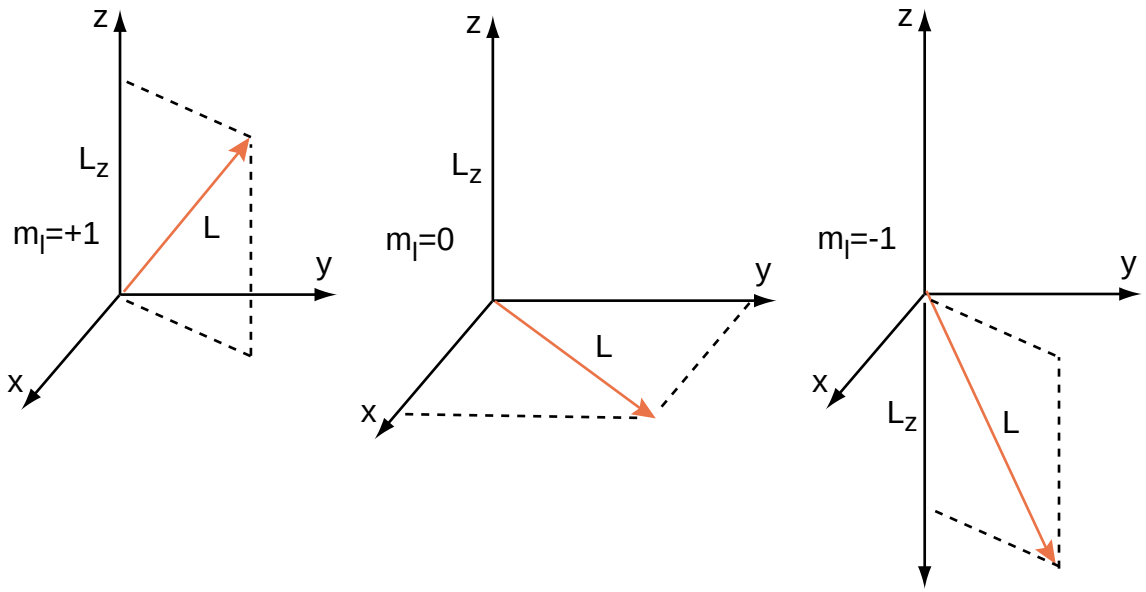

Figure 1: The configurations of the 3 possible angular momentum substates with $l = 1$.