PHY140Y

3 Properties of Gravitational Orbits

3.1 Overview

- Orbits and Total Energy
- Elliptical, Parabolic and Hyperbolic Orbits

3.2 Characterizing Orbits With Energy

Two point particles attracting each other via gravity will cause them to move, as dictated by Newton's laws of motion. If they start from rest, this motion will simply be along the line separating the two points. It is a one-dimensional problem, and easily solved. The two objects collide.

If the two objects are given some relative motion to start with, then it is easy to convince yourself that a collision is no longer possible (so long as these particles are point-like). In our discussion of orbits, we will only consider the special case where one object of mass M is much heavier than the smaller object of mass m. In this case, although the larger object is accelerated toward the smaller one, we will ignore that acceleration and assume that the heavier object is fixed in space.

There are three general categories of motion, when there is some relative motion, which we can characterize by the total energy of the object, *i.e.*,

$$\begin{aligned} E &\equiv K + U && (1) \\ &= \frac{1}{2}mv^2 - \frac{GMm}{r^2}, && (2) \end{aligned}$$

where K and U are the kinetic and gravitational energy, respectively.

3.3 Energy < 0

In this case, since the total energy is negative, it is not possible for the object to get too far from the mass M, since

$$\begin{aligned} E = K + U &< 0 && (3) \\ \Rightarrow \frac{1}{2}mv^2 &= \frac{GMm}{r} + E && (4) \\ \Rightarrow r &= \frac{GM}{\frac{1}{2}v^2 - \frac{E}{m}} && (5) \\ \Rightarrow r &< \frac{GM}{-E}, && (6) \end{aligned}$$

since r will take on its maximum value when the denominator is minimized (or when v is the smallest value possible, ie. zero). Remember, in this case E is negative! This implies that r can

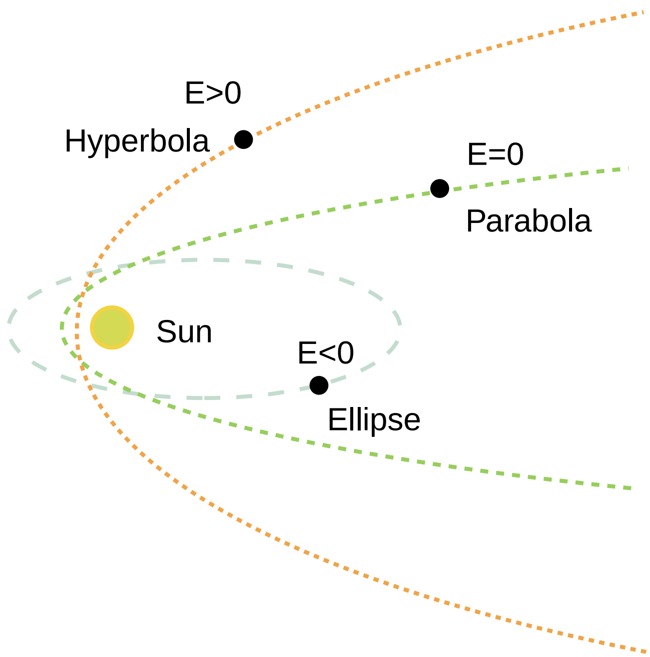

Figure 1: A comparison of the three different classes of gravitational orbits possible: elliptical closed orbits ($E = 0$), parabolic open orbits ($E = 0$), and hyperbolic open orbits ($E > 0$). Only the latter orbit is one where the object escapes the Sun's gravitational field.

never exceed a specific value and the orbit is therefore a **closed** one. This is illustrated in Fig. 1, where a typical elliptical orbit about the Sun is graphed.

If one performs a complete analysis of this (wait till second year applied mathematics or third year mechanics!), one finds that the possible solutions are in fact elliptical trajectories, characterized by a semimajor axis and an eccentricity. We will be interested here in one special class of such orbits – circular ones where the dynamics are relatively easy to understand.

3.4 Energy > 0

If the total energy E is greater than 0, then what can we say about the orbit? First, we note that because the gravitational potential energy is defined to be a negative quantity, this means that at all times the kinetic energy of the object is greater than zero, and that therefore it is always moving. This has the implication that the radius of the orbit is now unbounded and the orbit is not closed. Physically,

The actual trajectory of the orbit is another conic section – a hyberbola. It is in fact the trajectory taken by single-pass comets that come in from the Oort Cloud and have sufficient energy to escape the gravitational attraction of the Sun. A typical hyperbolic trajectory is shown in Fig. 1.

3.5 Energy $= 0$

In this case, the orbit is still not closed, as the object always has kinetic energy equal to the negative of its gravitational potential energy. Thus, it can only come to rest when it is an infinite distance away, ie. when $U \rightarrow 0$.

The trajectory of the object is yet a third type of conic section – a parabola – formed when you take a vertical slice out of a cone, and is illustrated in Fig. 1. Physically, this is an "open" orbit, since as the object recedes, it will continue to slow down but never actually stops. As the separation increases, both the potential energy and kinetic energy go to zero. This is in contrast to the case of a hyperbolic orbit, where as the potential energy goes to zero, the kinetic energy approaches a constant value that is greater than zero.

3.6 Gravitational Slingshots

An interesting application of hyperbolic orbits is their use in "gravitationally boosting" a deep-space probe. A good example of this is when the Pioneer probe was launched to Jupiter in 1989. It was first directed at Venus, where a hyperbolic orbit around Venus was used to place it into an elliptical orbit around the sun. This orbit was such that it would graze Earth about 10 months later. The resulting elliptical orbit had a period of exactly two years, so that it swung back and graze Earth a second time at exactly the same position, putting it on a high-speed intercept orbit with Jupiter 3 years later (see Figure 9-24 in **W&P**). The two-year orbit was designed to allow Pioneer to pay a visit to one of the more interesting Asteroid's, Gaspara.

In each "grazing" collision, momentum from the planet was transferred to the satellite, thereby increasing its radial velocity away from the sun. Since the planets are so much more massive than the probe, this additional energy comes for free.[1]

[1] What would happen if you tried to do this with just two smallest objects? Would it work?