PHY140Y

30 Addition of Angular Momentum

30.1 Overview

- Spin as Angular Momentum
- Rules for Adding Angular Momentum

30.2 Spin as Angular Momentum

With the recognition that the electron had an "intrinsic spin," the immediate question that arises is whether it is truly an angular momentum or something else. Our classical view of angular momentum arises from the view that it is always associated with one object spinning around another. A bare electron, having apparently no internal structure and no spatial extent, doesn't seem like a system that could have an angular momentum associated with it.

However, a more fundamental basis for angular momentum comes from looking at how systems behave when we rotate them. We find that when we look at systems with rotational symmetry about an axis, we must have associated with such systems a conserved quantity. That quantity is angular momentum. We thus find that elementary particles also have to have this sort of symmetry – that they have associated with them an angular momentum, or what we call an intrinsic spin.

In fact, when we look at these systems from a quantum mechanical perspective, we find that elementary particles come in two groups, with the first group having integer intrinsic spin (ie., $s = 0, 1, 2, \ldots$) and the second group having half-integral intrinsic spin (ie., $s = 1/2, 3/2, 5/2, \ldots$). We call the former particles "bosons" (in honour of the Indian physicist Bose) and the later group 'fermions" (in honour of the Italian physicist Fermi). We'll see later that fermions and bosons have rather different properties.

30.3 Rules for Adding Angular Momentum

We now have in a hydrogen atom two different sources of angular momentum, the orbital angular momentum $\vec{L}$, specified by the quantum numbers l and m_l, and the intrinsic spin of the electron $\vec{S}$, specified by $s = 1/2$ and m_s (the latter being either $+1/2$ or $-1/2$). Quantum mechanics respects the fact that a system with no external torques must conserve total angular momentum. However, it has very specific rules for how angular momentum can be added. In fact, quantum mechanics tells us that we have to define a new "total angular momentum" $\vec{J}$ that has the usual properties of an angular momentum in quantum mechanics, but arises from the different possible ways that $\vec{L}$ and $\vec{S}$ can be added together.

It is not correct to think that $\vec{J} = \vec{L} + \vec{S}$. We find that although this notion makes some sense, the final total angular momentum must be specified by a total angular momentum quantum

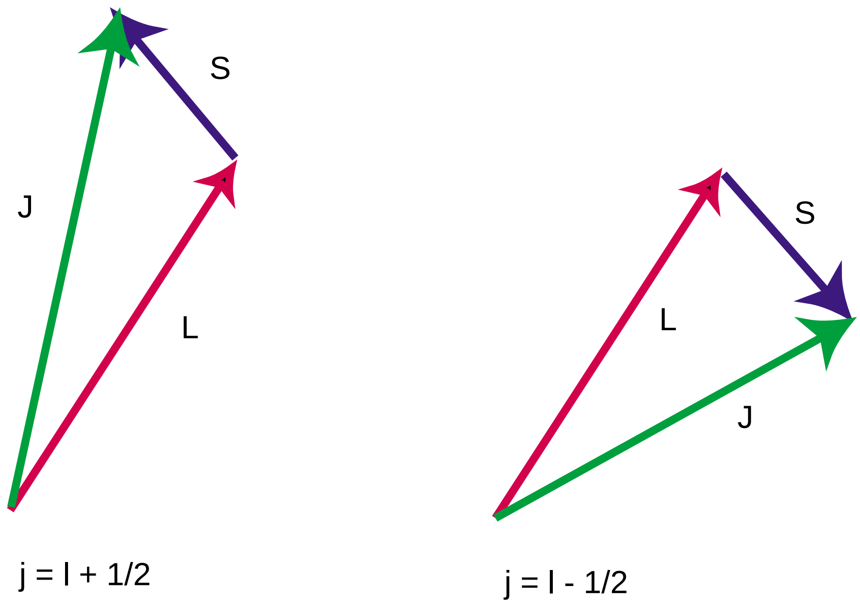

Figure 1: The "addition" of the orbital angular momentum and the intrinsic spin of the electron to produce the two possible values of total angular momentum.

number j and the projection of this angular momentum on $\hat{z}$, m_j. What we find is that for a given choice $\vec{L} \neq 0$ (and quantum number l), we can have two different total angular momentum states,

$$j = l + \frac{1}{2} \text{ and } j = l - \frac{1}{2}. \tag{1}$$

For $l = 0$, there is only one total angular momentum state, that with $j = s = 1/2$.

We can think of the two possible total angular momentum states as arising from the two possible ways the $\vec{L}$ and $\vec{S}$ vectors can be oriented with respect to each other. Although this is not a completely accurate model, it gives us some good intuitive feel for why the total angular momentum can take on two different values. We show in Fig. 1 the possible vectorial representations of the two ways we can construct the total angular momentum vector $\vec{J}$. The first possibility is one where $\vec{L}$ and $\vec{S}$ add together constructively to form a total angular momentum with $j = l + 1/2$. In that case, the length of the total angular momentum vector is

$$|\vec{J}| = \sqrt{(l + 1/2)(l + 3/2)}\hbar. \tag{2}$$

For this value of j, there are $2j + 1 = 2l + 2$ different quantum states specified by the magnetic angular momentum quantum number $m_j = -j, -j + 1, \ldots, j - 1, j$. The second possibility is one where $\vec{L}$ and $\vec{S}$ are in opposite directions. In this case, $j = l - 1/2$, and the total angular momentum has the magnitude

$$|\vec{J}| = \sqrt{(l - 1/2)(l + 1/2)}\hbar. \tag{3}$$

For this value of j, there are also $2j + 1 = 2l$ different quantum states defined again by the different values of m_j. Together, this gives us $4l + 2$ different quantum states.

We denote these states using the notation nX_j, where n is the principal quantum number, X is the spectroscopic notation for the value of l and j is the total angular momentum quantum number. So, for example, if $n = 2$ and $l = 1$, there are 4 quantum states with $2P_{3/2}$ specified by $m_j = -3/2, -1/2, 1/2, 3/2$ and there are 2 quantum states with $2P_{1/2}$, with $m_j = -1/2, 1/2$.