PHY140Y

4 Circular Orbits

4.1 Overview

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4.2 Circular Orbits

The special case of circular orbits is easy to analyze more completely. We know that to have an object moving with uniform circular motion of radius r_o and speed v_o, an object of mass m has to experience a centripetal acceleration with magnitude

$$a_c = \frac{v_o^2}{r_o}. \tag{1}$$

If this acceleration is provided by the force of gravity, then

$$\frac{GMm}{r_o^2} = \frac{mv_o^2}{r_o} \tag{2}$$

$$\Rightarrow v_o = \sqrt{\frac{GM}{r_o}}. \tag{3}$$

The period of the orbit T would be the time taken to travel the circumference, ie.

$$Tv_o = 2\pi r_o \tag{4}$$

$$\Rightarrow T = \frac{2\pi r_o}{v_o} \tag{5}$$

$$= \frac{2\pi r_o^{3/2}}{\sqrt{GM}} \tag{6}$$

$$\Rightarrow T^2 = \frac{4\pi^2}{GM} r_o^3, \tag{7}$$

which is Kepler's Third Law. Thus, we see at least in this case why Kepler would be correct. It turns out that this relationship is also true for elliptical orbits.

For circular orbits, the kinetic and potential energies are also constants of motion. In fact,

$$K = \frac{1}{2} m v_o^2 \tag{8}$$

$$= \frac{1}{2} m \left(\frac{GM}{r_o} \right) \tag{9}$$

$$= \frac{GMm}{2r_o}, \tag{10}$$

or exactly half of the negative of the gravitational potential energy. This means that the lower the orbit, the greater the kinetic energy and the greater the speed of the orbiting object. Yet the total energy is also lower!

4.2.1 Example: Space Shuttle Orbit

As a concrete example of what this implies, let's work out the velocity and period of a typical orbit of the space shuttle. It normally has a working orbit of around 700 km above the earth's surface, so

$$r_o = R_E + 7 \times 10^5 \text{ m} = 7.07 \times 10^6 \text{ m}. \tag{11}$$

Substituting that into the expressions for v_o above, we get

$$\begin{aligned} r_o &= \sqrt{\frac{GM}{r_o}} && (12) \\ &= \sqrt{\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})}{7.07 \times 10^6}} && (13) \\ &= 7.50 \times 10^3 \text{ m/s}. && (14) \end{aligned}$$

4.2.2 Example: Geosynchronous Satellites

The space shuttle's orbit is an example of a "low-earth" orbit, as it is not that far above the atmosphere. Geosynchronous satellites are those that always appear to stay over the same point on the earth's surface, ie. they are in orbits that have the same period as the earth, or $T = 24$ hr. These satellites are used almost always as communications satellites or as "death stars" in the TV broadcast industry.

We can start with Kepler's Third Law, and turn it around to express the radius of orbit in terms of the period. Thus,

$$\begin{aligned} T^2 &= \frac{4\pi^2}{GM} r_o^3 && (15) \\ \Rightarrow r_o &= \left(\frac{GMT^2}{4\pi^2} \right)^{1/3}. && (16) \end{aligned}$$

For a period $T = 24$ h $= 8.64 \times 10^4$ s, we find that the radius of the orbit is

$$\begin{aligned} r_o &= \left(\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})(8.64 \times 10^4)^2}{4\pi^2} \right)^{1/3} && (17) \\ &= 4.22 \times 10^7 \text{ m}. && (18) \end{aligned}$$

As this is the radius of the orbit as measured from the earth's centre, the height of a geosynchronous satellite above the earth's surface is

$$\begin{aligned} h &= r_o - R_E && (19) \\ &= 3.58 \times 10^7 \text{ m} = 35,800 \text{ km}. && (20) \end{aligned}$$

This is about 3 Earth "diameters" away, showing that such satellites are in fact quite far away. Nonetheless, they are still moving rapidly. The kinetic energy of a 100 kg sattelite is given by

$$\begin{aligned} K &= \frac{GMm}{2r_o} \qquad (21) \\ &= \frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})(100)}{2(4.22 \times 10^{7})} = 4.7 \times 10^{8} \text{ J}, \qquad (22) \end{aligned}$$

which is a lot of energy! All of this energy has to be provided by the booster rockets that put the satellite into orbit.