PHY140Y

5 Gravitational Fields

5.1 Overview

- Gravitational fields
- Gravitational Potential of a Spherical Object

5.2 Gravitational Fields

Given that the law of universal gravitation tells us what the gravitational force is everywhere, we can think of it as defining a “gravitational field,” namely a vector function $\vec{g}(\vec{r})$ of position in space. The actual form of the gravitational field associated with a mass M is

$$\vec{g}(\vec{r}) \quad = \quad \frac{GM}{r}\hat{r}, \tag{1}$$

where we have defined $\vec{r}$ in a coordinate system with the origin at the centre of the mass M. In this form, the law of universal gravitation becomes

$$\vec{F}(\vec{r}) \quad = \quad \vec{g}(\vec{r})m. \tag{2}$$

Thinking of gravity in terms of the gravitational field is convenient because it only depends on the mass generating the gravitational field.

On the Earth’s surface, given the size of the Earth, we tend to think of the gravitational field as always pointing downwards and having the same magnitude. In fact, this is not true in general, which leads to some interesting gravitational effects. The most intriguing is what we call “tidal forces” (see next lecture and this week’s discussion problems).

Suppose we have a “large” object V with a constant mass density ρ. We can calculate the gravitational field created by V by breaking it into small volume elements $\delta V_i, i = 1$ to N, and considering the gravitational field created by each volume element. Now we have to account for the fact that the mass element is at some position $\vec{r_i}$ so that the vector between the mass and the point in space is $\vec{r} - \vec{r_i}$. The total gravitational field is then the sum of the individual contributions, ie.

$$\vec{g}_{net}(\vec{r}) \quad = \quad \sum_{i=1}^{N} -\frac{G\rho\delta V_i}{|\vec{r} - \vec{r_i}|^3}(\vec{r} - \vec{r_i}) \tag{3}$$

$$\rightarrow \quad \int_V \frac{G\rho}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')\, dV', \tag{4}$$

where we have taken the limit where $\delta V \rightarrow dV'$ and turned the sum into a volume integral. The variable of integration is $\vec{r}'$. This is a generalized form of the law of universal gravity and allows

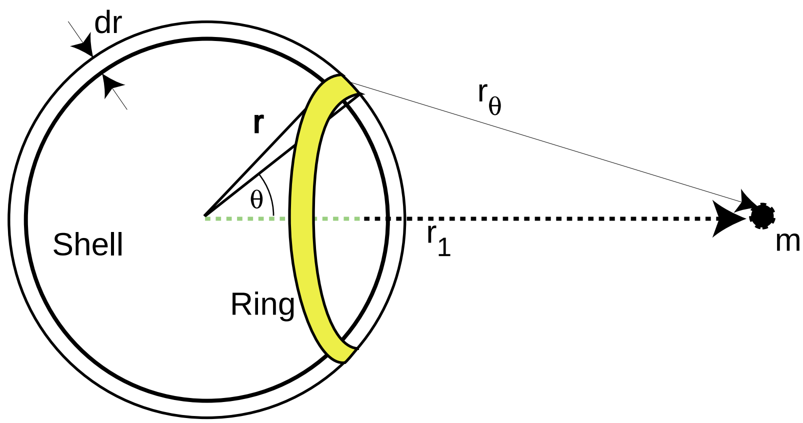

Figure 1: The calculation of the gravitational potential associated with a spherical mass. The mass has been broken down into "shells" of radius r with thickness dr. The angular interval $(\theta, \theta + d\theta)$ defines a ring of radius $r \sin\theta$ and width $r\, d\theta$.

one to calculate the gravity arising from a mass distribution. In fact, if V has a non-uniform mass density $\rho(\vec{r})$, the gravitational field becomes

$$\vec{g}_{net}(\vec{r}) \quad = \quad \int_V \frac{G\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')\, dV'. \tag{5}$$

In the same way, the gravitational potential for a mass distribution V can be written as

$$U(\vec{r}) \quad = \quad \int_V \frac{G\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}\, dV'. \tag{6}$$

It turns out that in many problems, this expression for U is a much more convenient way of solving for the effects of gravity. The primary thing to note is that U is just a scalar function of $\vec{r}$ whereas $\vec{g}(\vec{r})$ is in fact a vector function and so involves in principle three times the work to sort out.

5.2.1 Spherical Mass Distributions

As a concrete example of this, let's work out the gravitational field associated with a spherically symmetric mass distribution. In this case, it is far easier to work out the gravitational potential of a point mass m at the point $\vec{r}$, and then solve for the gravitational field.

Suppose we have a uniform spherical mass M with radius R and we want to calculate the effects of gravity at a distance $\vec{r_1}$, as shown in Fig. 1. To solve this problem, we first recognize that the problem has axial symmetry, so we choose to work in a spherical-polar coordinate system with the $\hat{z}$ axis along the direction defined by $\vec{r_1}$.

We then break down the spherical mass into shells of thickness dr. If θ is the polar angle, then all the mass in a ring defined by the angular interval between θ and $\theta + d\theta$ is at the same distance from the point $\vec{r_1}$. We will call this distance r_θ (since it depends on θ). This distance is given by

$$r_\theta \quad = \quad \sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}, \tag{7}$$

using the cosine law. The mass in this "ring", dM, is just the surface area of the ring ($2\pi r\sin\theta \times r d\theta$) times its thickness ($dr$) times its density ρ. Thus the gravitational potential energy at the point $\vec{r_1}$ due to the ring is

$$\begin{aligned} dU_r &= -\frac{GdM}{r_\theta} && (8)\\ &= -\frac{G\rho(2\pi r\sin\theta\, rd\theta\, dr)}{\sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}} && (9)\\ &= -2\pi G\rho r^2 dr\frac{\sin\theta d\theta}{\sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}}. && (10)\end{aligned}$$

In order to sum up the contributions from the entire shell, we now integrate this expression with respect to θ from 0 to π, effectively adding up all the possible rings that make up the surface of the shell. This gives

$$U_r(\vec{r_1}) \quad = \quad -2\pi G\rho r^2 dr\int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}}. \tag{11}$$

We can do the integral easily by transforming from the variable θ to $\eta \equiv \cos\theta$. Then

$$\begin{aligned} d\eta &= -\sin\theta d\theta && (12)\\ \Rightarrow \int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}} &= \int_1^{-1} \frac{-d\eta}{\sqrt{r_1^2 + r^2 - 2rr_1\eta}} && (13)\\ &= \frac{-1}{rr_1}\left[\sqrt{r_1^2 + r^2 - 2rr_1} - \sqrt{r_1^2 + r^2 + 2rr_1}\right] && (14)\\ &= \frac{2}{r_1}. && (15)\end{aligned}$$

So, with this, we get the potential for the shell to be

$$\begin{aligned} U_r(\vec{r_1}) &= -\frac{G\left(4\pi r^2 dr\right)\,\rho}{r_1} && (16)\\ &= -\frac{GM_r}{r_i}, && (17)\end{aligned}$$

where we have taken the mass of the shell, M_r, to be equal to its surface area times its thickness times its density.

Well, I claim we are now done. What we have already shown is that the gravitational potential energy of a "shell" of mass is equivalent to the situation where all the mass is at the point of symmetry, or the origin in this case. Since the total sphere is made up of just a series of shells of this sort, the total gravitational potential energy associated with the entire sphere is

$$U(\vec{r_1}) \quad = \quad -\frac{GM}{r_1}. \tag{18}$$

But we haven't worked out the gravitational field! This is "trivial" to do using the relationship

$$\vec{F}(\vec{r}) = -\Delta U, \tag{19}$$

where Δ is the gradient operator, ie.

$$F_x = -\frac{\delta U}{dx} \tag{20}$$

$$F_y = -\frac{\delta U}{dy} \tag{21}$$

$$F_z = -\frac{\delta U}{dz}. \tag{22}$$

Try it!

Finally, what would be the gravitational field **inside** the shell? A similar calculation will show you that the field is zero. Although this may not seem that intuitive, think about it a little and try to understand why this is something that you might have expected.