PHY140Y

7 Electrostatics and Coulomb's Law

7.1 Overview

- Electrostatics
- Coulomb's Law
- Electric Field
- Superposition Principle
- Continuous Charge Distribution

7.2 Electrostatics

Electricity and magnetism comprise a host of phenomena that have an enormous impact on our environment. Perhaps as importantly, we have been able to exploit these phenomena to form the basis for perhaps the most conveneient means to distribute and direct energy.

There has been a recorded awareness of these phenomena for close to three millenia, although aspects of electricity (lightening and thunder are good examples) surely were evident to man from the very earliest times. The basic principles stem from the forces and dynamics associated with charges, both static and flowing. Static charge distributions create electrostatic forces, whereas flowing charges (or "currents") are responsible for both electric and magnetic forces. We will limit our focus to the basics of electrostatics, leaving the richer and more complex phenomena of electromagnetism for later study.

7.3 Coulomb's Law

The concept of "electric charge" was introduced to help explain the existence of the electromagnetic force. Early studies suggested that two different types of electric charge exist, which we label as "positive" and "negative." These studies also showed that the force exerted on two electric charges, q_1 and q_2, separated by a distance $\vec{r}$, is given by

$$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}, \tag{1}$$

where we quantify the unit of charge in Coulombs, k is a constant with value 9×10^9 Nm^2/C^2, and the direction of the force is given by the unit vector $\hat{r}$. This is known as Coulomb's Law.

A Coulomb is in practice a very large amount of charge. Charge is quantized, with the fundamental unit being the charge of an electron. This "elementary charge" has a value of -1.60×10^{-19} C. Note that it has a negative value. The proton charge is opposite in sign to that of the electron, but has exactly the same magnitude.

To get a feeling for what a Coulomb is, let's go back to the example of the two charged pith balls we discussed in the first lecture when we compared the electrostatic and gravitational forces. In that example, after we had charged both pith balls with what we assumed were equal charges q[1], we found the electric force to have magnitude

$$F_E = 1.3 \times 10^{-5} \text{ N}. \tag{2}$$

Since the balls were separated by $r = 2$ cm, we now know from Coulomb's Law,

$$\begin{aligned} F_E &= \frac{kq^2}{r^2} && (3) \\ \Rightarrow q &= \sqrt{\frac{F_E r^2}{k}} && (4) \\ &= \sqrt{\frac{(1.3 \times 10^{-5})(0.02)^2}{9.0 \times 10^9}} = 8 \times 10^{-10} \text{ C}. && (5) \end{aligned}$$

So the pith balls each carried only about 1 billionth of a Couloumb. On the other hand, this corresponds to of order 10^{10} elementary charges.

7.4 Electric Field

Given a point charge q_1 located at the origin, it is convenient to rewrite Coulomb's Law in the following way for the force on a charge q_2 at any other point $\vec{r}$:

$$\begin{aligned} \vec{F}(\vec{r}) &= \frac{kq_1q_2}{r^2}\hat{r} && (6) \\ &= q_2\left(\frac{kq_1}{r^2}\hat{r}\right) && (7) \\ &= q_2\,\vec{E}\,(\vec{r})\,, && (8) \end{aligned}$$

where we have now introduced the **electric field**, $\vec{E}(\vec{r})$, which is a vector function of space. It has units of N/C, and it is analogous (though not equivalent) to the gravitational acceleration introduced when we studied gravity. It's primary convenience is that we can now talk about the effects that a single charge or a set of charges could have at any point in space without having to state up-front what charge is placed at that point.

For example, returning to our over-used set of pith balls, we can now state that the magnitude of the electric field created by one of the pith balls at the location of the other is

$$\begin{aligned} |\vec{E}| &= \frac{kq_p}{r^2} && (9) \\ &= \frac{(9.0 \times 10^9)(8 \times 10^{-10})}{(0.02)^2} = 1.8 \times 10^4 \text{ N/C}. && (10) \end{aligned}$$

So electric fields can take on rather large values. Within a thunderstorm, for example, electric fields of order 500 000 N/C or more are typical.

[1] What evidence do we have to support the assumption that the pith balls are uniformly charged?

Although we won't need it till next lecture, it makes sense to introduce a new unit, the **Volt**, which is defined as 1 Joule per Coulomb. Since a Joule is a Newton-metre, the unit N/C is equivalent to Volt/m and this latter quantity is often the unit used in discussions of electric fields (remember that we capitalize any units that derive from proper names!).

7.5 Principle of Superposition

The electrostatic force between two point charges described by Coulomb's Law applies to more complex charge distributions. In particular, if we have three charges, q_1, q_2 and q_3, located at positions $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, the force on charge q_3 is just the sum of the force applied to q_3 due to q_1 and the force applied to q_3 due to q_2. The total force is

$$\vec{F}_3 = \frac{kq_1q_3}{r_{13}^2}\hat{r}_{13} + \frac{kq_2q_3}{r_{23}^2}\hat{r}_{23}, \tag{11}$$

where $\hat{r}_{13}$ and $\hat{r}_{23}$ are the unit vectors in the directions of $\vec{r}_3 - \vec{r}_1$ and $\vec{r}_3 - \vec{r}_2$, respectively.

As a concrete example, consider a situation where we have a raindrop with charge $+q$ placed on the x-axis at a position $x = -a$, and a second raindrop with charge $+q$ placed on the same axis at $x = a$, as shown in Fig. 1. We wish to calculate the electric field at a point y along the y-axis. The electric fields of each raindrop can be written as

$$\vec{E}_- = \frac{kq}{a^2 + y^2}\left(\sin\theta\hat{x} + \cos\theta\hat{y}\right) \tag{12}$$

$$\vec{E}_+ = \frac{kq}{a^2 + y^2}\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right), \tag{13}$$

where $\vec{E}_-$ and $\vec{E}_+$ refer to the electric fields of the raindrop at $x = -a$ and $x = +a$, respectively. When we add these together using the principle of superposition, we find the total electric field to be

$$\vec{E}_{tot} = \vec{E}_- + \vec{E}_+ \tag{14}$$

$$= \frac{kq}{a^2 + y^2}\,2\cos\theta\hat{y}. \tag{15}$$

The angle θ is between the electric field and the y-axis, as shown in Fig. 1, and from the geometry illustrated in the figure, we see that

$$\cos\theta = \frac{y}{\sqrt{a^2 + y^2}}. \tag{16}$$

Thus, the total electric field is

$$\vec{E}_{tot} = \vec{E}_- + \vec{E}_+ \tag{17}$$

$$= \frac{kq}{a^2 + y^2}\,2\frac{y}{\sqrt{a^2 + y^2}}\hat{y} \tag{18}$$

$$= \frac{2kqy}{(a^2 + y^2)^{3/2}}\hat{y}. \tag{19}$$

We see that it is only in the $\hat{y}$ direction, as we might have guessed from the symmetry of the problem.

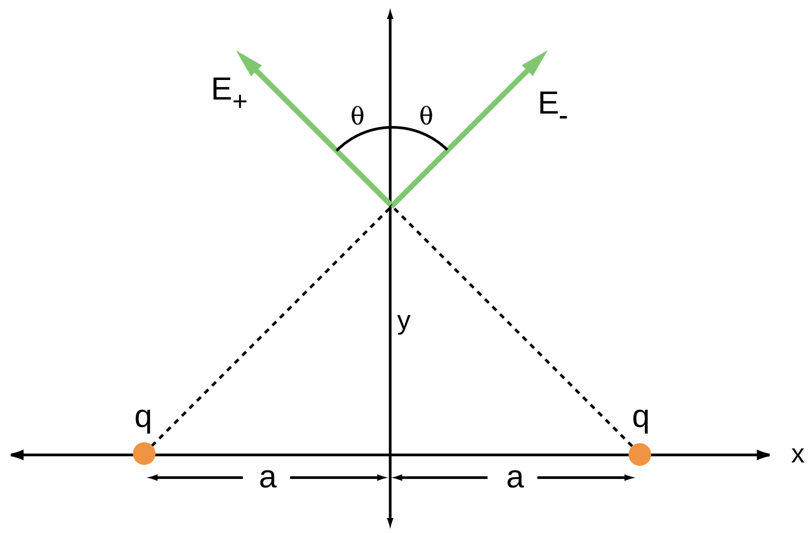

Figure 1: Two charged raindrops placed on the x-axis, and the calculation of the corresponding electric field at a point on the y-axis.

7.6 Continuous Charge Distributions

Our discussion so far has been in the context of point charges, which may make physical sense since nature seems to provide charges in discrete units, with the smallest unit of charge being the charge found on an electron or proton. However, this unit of charge is so small and the number of electrons and protons so enormous that we often find it more convenient to think of charge as being distributed continuously over a volume.

We therefore often talk about a continuous charge distribution, and can associate with to each charge distribution a charge density function, which historically has been labelled by the Greek letter ρ. The charge density of a given material is viewed as being a function of space, $\rho(\vec{r})$. For sake of argument, suppose we have a volume V that has a continuous charge distribution in it. Let's subdivide the volume V into a finite number, N, of small volume elements dV_i, where each volume element is represented by a vector $\vec{r_i}$ that locates the centre of the volume element. We will define dV_i to be equal to the size of the volume element (so it has units of m^3).

With this notation, we can now address the question of what form the electric field would take for a charge distribution in the volume V. The electric field at a point $\vec{r}$ arising from the charge in a volume element dV_i is, from Coulomb's Law,

$$d\vec{E}_i = \frac{k\,\rho(\vec{r_i})dV_i}{|\vec{r}-\vec{r_i}|^2}\,\frac{\vec{r}-\vec{r_i}}{|\vec{r}-\vec{r_i}|}. \tag{20}$$

The unit vector is in the direction of the displacement between the point where we are evaluating the electric field and the point identifying the location of the volume element. By superposition, the total electric field is

$$\vec{E}(\vec{r}) = \sum_i^N \frac{k\,\rho(\vec{r_i})dV_i}{|\vec{r}-\vec{r_i}|^2}\,\frac{\vec{r}-\vec{r_i}}{\vec{r}-\vec{r_i}} \tag{21}$$

$$\rightarrow \int_V \frac{k\,\rho(\vec{r}_\prime)}{|\vec{r}-\vec{r'}|^2}\,\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|}\,dV', \tag{22}$$

where in the last step, we have shrunk the size of the volume elements to zero (in principle, increasing the number of volume elements) and turned the sum into a three-dimensional volume integral. In this notation, the infinitesimal $dV = dx\,dy\,dz$ in Cartesian coordinates.

7.6.1 The Electric Field of a Power Line

As a concrete example, let's consider a very long power line with cross-sectional area $A = 1\ \text{cm}^2 = 1\times10^{-4}\ \text{m}^2$ carrying a uniform charge distribution $\rho = 1\times10^{-1}\ \text{C/m}^3$. We can assume that the conductor is so thin, that in fact all of the charge lies on a line along the centre of the conductor.

In this case, the charge density is really only a function of position along the conductor. We can see this explictly by noting that in this problem

$$\begin{aligned} dV &= A\,dx & (23)\\ \Rightarrow \rho\,dV &= \rho A\,dx & (24)\\ &= (\rho A)\ dx & (25)\\ &= \lambda\,dx, & (26)\end{aligned}$$

where λ is now a linear charge density with value

$$\begin{aligned} \lambda &= \rho A & (27)\\ &= (1\times10^{-1})(1\times10^{-4}) = 1\times10^{-5}\ \text{C/m}. & (28)\end{aligned}$$

We want to calculate the electric field at a point $y = 30$ m away from the power line.

To do this problem, we choose a coordinate system so that we can place the power line on the x-axis and the point where we want to determine the electric field on the y-axis. We see that there is a left-right symmetry in this problem: If we look at the charge in a small element dx along the x-axis at a point x, there is a corresponding point at $-x$ where the contribution to the electric field has the same magnitude, but whose component along the $\hat{x}$ direction has the opposite sign. If $\lambda\,dx$ is the charge at the point x, then the contribution to the electric field from these two charge elements is (from the earlier charged raindrop problem)

$$d\vec{E}(y) = \frac{2ky\hat{y}}{(x^2+y^2)^{3/2}}\lambda\,dx. \tag{29}$$

The total electric field would be the integral of this expression from $x = 0$ to $x = \infty$ (remember we have accounted for the negative x values already). Thus

$$\vec{E}(y) = \int_0^\infty \frac{2ky\hat{y}}{(x^2+y^2)^{3/2}}\lambda\,dx. \tag{30}$$

I don't expect you to have to figure out this integral. You can find it in various tables of integrals. The result is

$$\begin{aligned}
\vec{E}(y) &= \int_0^\infty \frac{2ky\hat{y}}{(x^2+y^2)^{3/2}}\lambda\, dx && (31)\\
&= 2k\lambda y \left[\frac{x}{y^2\sqrt{x^2+y^2}}\right]_{x=0}^{x=\infty} \hat{y} && (32)\\
&= \frac{2k\lambda}{y}\hat{y}. && (33)
\end{aligned}$$

We see that at 30 m away from the power line, the electric field would be pointed away from the power line and would have magnitude

$$E(\vec{r}) = \frac{2(1\times 10^{-5})(9\times 10^{9})}{30} = 6 \text{ kV/m}. \qquad (34)$$

Notice that the electric field falls off as $1/y$, much more slowly than a point charge, and points away from the power line.