PHY140Y

9 Using Gauss's Law

9.1 Overview

- Gauss's Law
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9.2 Gauss's Law

9.3 Applications of Gauss's Law

Coulomb's Law has several implications. It first tells us that electric charges are "sources" of the electric field. It also tells us that the Coulomb force is very long-range, given the $1/r^2$ behaviour of the force. Together, these two aspects give the electric field an unusual property, which is encapsulated in Gauss's Law.

First, let's go back to the simple case of a single point charge q. Let's place an imaginary sphere around this charge at a radius r and let's now look at the surface of this sphere. We can divide up the sphere's surface S into a large number of small "patches" or elements, which I will denote by dS_i, where the value of dS_i is the actual area of the ith small patch. Each patch, provided it is small enough, can be approximated as a flat tile so that we can define the unit vector $\hat{n}_i$ as the vector normal to the surface dS_i, as shown in Fig. 1. To make the notation more compact, I will denote the product $dS_i\hat{n} = d\vec{A}_i$.

Now at each patch $d\vec{A}_i$, let $\vec{E}_i$ be the electric field vector at the centre of the patch. We then define the **flux** of the electric field through the surface element to be

$$d\phi_i \equiv \vec{E}_i \cdot d\vec{A}_i. \tag{1}$$

The flux is a scalar quantity, and one can think of it as representing the "flow" of the electric field through the surface element. However, it does not have a real physical interpretation as a flow, since there really is nothing flowing. However, it is a convenient picture.

For a surface element on the imaginary sphere surrounding the charge q, we can readily calculate the flux through any surface element. We note that the electric field at a distance r from the charge is

$$\vec{E} = \frac{kq}{r^2}\,\hat{r}, \tag{2}$$

while the surface element of a patch of the sphere would have a normal pointing in the $\hat{r}$ direction. This means that

$$d\phi_i = \vec{E}_i \cdot d\vec{A}_i \tag{3}$$

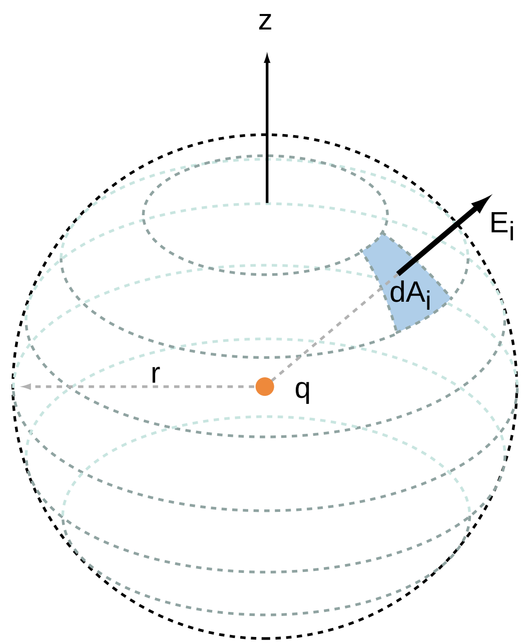

Figure 1: The calculation of the surface integral of a sphere centred on the charge q. The imaginary sphere of radius r is centred on the point charge. The surface element dA_i has a normal to it that is parallel to the electric field $\vec{E}_i$ at the centre of the patch.

$$
\begin{aligned}
&= \frac{kq}{r^2}\hat{r} \cdot dS_i \hat{r} && (4)\\
&= \frac{kq}{r^2}\, dS_i. && (5)
\end{aligned}
$$

We see that the ratio on the right-hand side is a constant at any point on the sphere.

We can then define the total flux of the electric field through the surface S to be

$$
\begin{aligned}
\phi &= \sum_i d\phi_i && (6)\\
&= \sum_i \vec{E}_i \cdot d\vec{A}_i && (7)\\
&\to \int_S \vec{E} \cdot d\vec{A}, && (8)
\end{aligned}
$$

where we have let the size of each surface element become infinitesimally small and form a two-dimensional **surface integral**. We can calculate the surface integral by introducing spherical-polar angular coordinates, θ' and ϕ', in which case the surface element becomes $dS = r\sin\theta'\, r d\phi'$:

$$
\begin{aligned}
\phi &= \int_S \vec{E} \cdot d\vec{A} && (9)\\
&= \int_0^{2\pi}\int_0^{\pi} \frac{kq}{r^2} r^2 \sin\theta'\, d\theta'\, d\phi' && (10)\\
&= \frac{kq}{r^2}\, r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta'\, d\theta'\, d\phi' && (11)\\
&= 4\pi kq, && (12)
\end{aligned}
$$

since the integral is now just the total surface area of a sphere with unit radius.

We see that the total flux through this imaginary surface only depends on the charge inside the volume defined by the surface, and doesn't depend on the radius of the sphere. This is called Gauss's Law.

9.3.1 Gauss's Law for Arbitrary Surfaces

The result obtained in the previous section holds not only for symmetric surfaces, but for any arbitrary surface S that contains a volume V. Furthermore, this result can be extended to arbitrary charge distributions inside the volume.

To express what is really going on, we can write

$$\phi \equiv \int_S \vec{E} \cdot d\vec{A} \tag{13}$$

$$= 4\pi k \int_V \rho(\vec{r}\,') \, dV', \tag{14}$$

where we have used $\rho(\vec{r})$ to denote the charge density in the volume V. In effect, the right-hand side of Eq. 14 counts up the total charge inside the volume V.

There is one minor wrinkle to this, which is that normally Gauss's Law is expressed in terms of the **permittivity** ϵ_0, which is defined as

$$\epsilon_0 \equiv \frac{1}{4\pi k} = 8.85 \times 10^{-12} \ \mathrm{C}^2/\mathrm{N}\,\mathrm{m}^2. \tag{15}$$

With this definition, we can write

$$\phi \equiv \int_S \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int_V \rho(\vec{r}\,') \, dV' \tag{16}$$

Gauss's Law could be considered a curiousity of the electric field if it were not such a useful tool for calculating electric fields or charge distributions in cases where there is enough symmetry. We will work a few examples to see how this works.

9.3.2 Point Charge

Let's return to a point charge q placed at the origin. We will use Gauss's Law to determine the electric field anywhere.

We first note that this problem has a spherical symmetry, so that the electric field must be proportional to $\hat{r}$ everywhere. Furthermore, at a given distance r, the electric field would have to have equal strength, $|\vec{E}(\vec{r})|$, regardless of the which direction $\vec{r}$ is pointing from the charge.

This, the electric flux through an imaginary sphere drawn around the charge with radius r would be

$$\phi = \int_S \vec{E} \cdot d\vec{A} \tag{17}$$

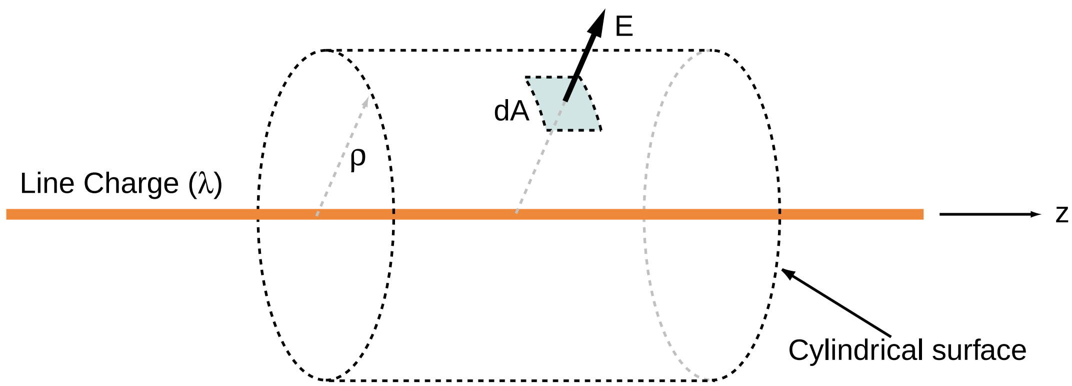

Figure 2: A linear charge distribution, and the imaginary "Gaussian" cylinder used to apply Gauss's Law. The small surface element shown here as dA has a normal that is parallel to the electric field at the centre to the patch (shown as $\vec{E}$).

$$
\begin{aligned}
&= |\vec{E}(r)| \int_S dS && (18)\\
&= 4\pi r^2 |\vec{E}(r)| && (19)\\
&= \frac{1}{\epsilon_0} \int_V \rho(\vec{r}') \, dV' && (20)\\
&= \frac{q}{\epsilon_0} && (21)\\
\Rightarrow |\vec{E}(r)| &= \frac{q}{4\pi\epsilon_0 r^2}, && (22)
\end{aligned}
$$

which if you look closely is just Coulomb's Law for a point charge. Note that we will often use the term "Gaussian" surface to identify these imaginary surfaces that are used to apply Gauss's Law.

9.3.3 Line Charge

A similar argument holds for a linear charge distribution. Suppose we have a long straight wire sitting on the $\hat{z}$ axis with a charge distributed on it so that it has λ charge per unit length.

First, we note that by symmetry, the electric field can only be perpendicular to the line charge. Thus the ensuing electric field has **cylindrical** symmetry. The natural coordinate system for this problem is cylindrical: Relative to a Cartesian coordinate system defined by a set of $\hat{x}$, $\hat{y}$ and $\hat{z}$ axes, any point in space can also be described by (ρ, ϕ, z), where ρ is the distance from the $\hat{z}$ axis, ϕ is the azimuthal angle of the point in the $\hat{x} - \hat{y}$ plane, and z is the position along the $\hat{z}$ axis. In this coordinate system, the electric field would always point in the $\hat{\rho}$ direction.[1]

[1] Don't confuse the coordinate ρ with the charge density function $\rho(\vec{r})$. Where any confusion is possible, I will make the distinction between these concepts by always noting that charge density to be a function of space.

To take advantage of this symmetry, we create an imaginary "Gaussian" cylindrical surface of radius ρ and length L, as shown in Fig. 2. The electric field, being in the direction perpendicular to the $\hat{z}$, is now perpendicular to the normal to the surface at each end of the cylinder, so that the dot product $\vec{E} \cdot d\vec{A}$ on the ends of the cylindrical surface are zero. Thus, the flux of the electric field through the cylinder's surface is

$$\begin{aligned} \phi &= \int_S \vec{E} \cdot d\vec{A} && (23) \\ &= \int_{sides} E(\rho)\hat{\rho} \cdot \hat{\rho}\, dS && (24) \\ &= E(\rho) \int_{sides} dS && (25) \\ &= E(\rho) 2\pi\rho L, && (26) \end{aligned}$$

where the integral in Eq. 25 is just the area of the cylinder, or $2\pi\rho L$. The charge enclosed in the cylinder is just the length of the line charge in the cylinder times the linear charge density λ. Thus, Gauss's Law tells us

$$\begin{aligned} \phi = E(\rho) 2\pi\rho L &= \frac{1}{\epsilon_0}\lambda L && (27) \\ \Rightarrow E(\rho) &= \frac{\lambda}{2\pi\epsilon_0\rho}. && (28) \end{aligned}$$

Note that the electric field falls off as $1/\rho$ instead of as $1/r^2$ in the case of the point charge.