PHY1510H

Solutions to Problem Set 1

4 October 2010

1. (a) One simply grinds through the calculations of the divergence and curl of the vector field. For convenience, we use a spherical-polar coordinate system with charge q_2 at the origin. Since the electric field now only has radial dependence, we find

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) && (1)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 q_2 r^{-\delta}\right) && (2)\\ &= \frac{-q_2\delta}{r^{3+\delta}}. && (3) \end{aligned}$$

Similarly,

$$\begin{aligned} \nabla \times \vec{E} &= \hat{r}\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi) - \frac{\partial E_\theta}{\partial\phi}\right] + \hat{\theta}\frac{1}{r\sin\theta}\left[\frac{\partial E_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial\phi}(rE_\phi)\right] + && (4)\\ &\quad \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right] && (5)\\ &= 0 && (6) \end{aligned}$$

as only E_r is non-zero and only depends on r.

(b) We note that since the two shells are connected with a conducting wire, they must be at the same potential; otherwise, charge would flow from one shell to the other. The potential energy of a charge q_1 in an electric field generated by charge q_2 at the origin is the negative of the work done to bring that charge from infinity. Hence

$$\begin{aligned} \Phi(r_1) &= -\int_\infty^{r_1}\left(\vec{E}\cdot\hat{r}\right)dr && (7)\\ &= -\int_\infty^{r_1}\left(\frac{kq_1}{r^{2+\delta}}\right)dr && (8)\\ &= \left(\frac{1}{1+\delta}\right)\left(\frac{kq_1}{r_1^{1+\delta}}\right). && (9) \end{aligned}$$

We can now calculate the potential of a charge q located a distance r from the centre of a conducting sphere of radius b and with charge q_b centred at the origin. We choose a polar coordinate system with the z-axis passing through the location of charge q. We divide the sphere into annuli defined by $(\theta, \theta + d\theta)$, where θ is the polar angle. The distance of all the charge on this annulus to charge q is

$$\sqrt{r^2 + b^2 - 2rb\cos\theta}, \qquad (10)$$

by the cosine law. The total charge on this belt is given by the surface charge density

$$\sigma_b = \frac{q_b}{4\pi b^2}, \tag{11}$$

multiplied by the area of the annulus, $2\pi\ b^2 \sin\theta d\theta$, or

$$\begin{aligned} dq &= \frac{q_b}{4\pi b^2} \times 2\pi\ b^2 \sin\theta d\theta &(12)\\ &= \frac{q_b \sin\theta}{2}\, d\theta. &(13) \end{aligned}$$

The contribution of this annulus to the potential for the charge is then

$$d\Phi = \left(\frac{1}{1+\delta}\right)\left(\frac{kq_b}{2\left(r^2+b^2-2rb\cos\theta\right)^{(1+\delta)/2}}\right)\sin\theta d\theta. \tag{14}$$

We now integrate over all θ to obtain

$$\begin{aligned} \Phi_b(r) &= \int_0^\pi \left(\frac{1}{1+\delta}\right)\left(\frac{kq_b}{2\left(r^2+b^2-2rb\cos\theta\right)^{(1+\delta)/2}}\right)\sin\theta d\theta &(15)\\ &= \frac{q_b}{2rb\left(1-\delta^2\right)}\left[\left(r^2+b^2+2rb\right)^{(1-\delta)/2} - \left(r^2+b^2-2rb\right)^{(1-\delta)/2}\right]. &(16) \end{aligned}$$

This form is true both inside and outside the sphere.
We can similarly calculate the potential arising from the charge on the sphere with radius a and charge q_a and it is the same form as Eq. 16 replacing $a \leftrightarrow b$. The potential at each sphere is the sum of these contributions, and have to be equal. Hence,

$$\begin{aligned} \Phi_a(a)+\Phi_b(a) &= \Phi_a(b)+\Phi_b(b) &(17)\\ \Rightarrow \frac{q_a\left(2a\right)^{-\delta}}{a\left(1-\delta^2\right)} + \frac{q_b\left(2ab\right)^{-\delta}}{a\left(1-\delta^2\right)}\Big[\left(a+b\right)^{1-\delta} &- \left(a-b\right)^{1-\delta}\Big] = \\ \frac{q_a\left(2b\right)^{-\delta}}{b\left(1-\delta^2\right)} &+ \frac{q_b\left(2ab\right)^{-\delta}}{b\left(1-\delta^2\right)}\left[\left(a+b\right)^{1-\delta}-\left(a-b\right)^{1-\delta}\right] \\ \Rightarrow \frac{q_b}{2}\Big[\left(a+b\right)^{1-\delta}-\left(a-b\right)^{1-\delta} &- 2^{1-\delta}ab^{-\delta}\Big] = \\ \frac{q_a}{2}\Big[\left(a+b\right)^{1-\delta} &- \left(a-b\right)^{1-\delta}-2^{1-\delta}ba^{-\delta}\Big]. &(18) \end{aligned}$$

To further simplify this, we assume $\delta << 1$ and use the Taylor series expansion for the exponential function

$$\begin{aligned} f(x) &\equiv p^x = e^{x\ln p} &(19)\\ \Rightarrow \frac{df}{dx} &= \ln p\, p^x &(20)\\ \Rightarrow f(x+\Delta x) &\simeq p^x + p^x \ln p\, \Delta x. &(21) \end{aligned}$$

Applying that to the first two terms in the square brackets above on the LHS, we find

$$\left(a+b\right)^{1-\delta}-\left(a-b\right)^{1-\delta} \simeq 2b - \delta\left[a\ln\left(\frac{a+b}{a-b}\right) + b\ln\left(a^2-b^2\right)\right]. \tag{22}$$

Similarly, we can expand

$$b^{-\delta} \simeq 1 - \delta \ln b \quad \text{and} \quad a^{-\delta} \simeq 1 - \delta \ln a. \tag{23}$$

Putting all this together, and defining the constant

$$c \equiv a \ln \left(\frac{a+b}{a-b}\right) + b \ln \left(a^2 - b^2\right), \tag{24}$$

we obtain the relationship

$$\begin{aligned} q_b \left[2b - c\delta - 2^{1-\delta} a \left(1 - \delta \ln b\right)\right] &\simeq q_a \left[2b - c\delta - 2^{1-\delta} b \left(1 - \delta \ln a\right)\right] & (25) \\ \Rightarrow q_b &\simeq -q_a \delta \left[\frac{2b \ln(2a) - c}{2(a-b)}\right] & (26) \end{aligned}$$

to lowest order in δ.

(c) If δ is zero, we know from Gauss's Law that q_b must be zero, since that would be the only way to have the two spheres at constant potential. From the above result, we now have an expression to lowest order in δ of the residual charge on sphere b that must accumulate if we place a charge on sphere a:

$$\delta \simeq -\frac{q_b}{q_a} \left[\frac{2(b-a)}{2b \ln(2a) - c}\right]. \tag{27}$$

Hence, the experiment to be done is to place the charge on sphere a, then disconnect the two spheres electrically and measure the charge on sphere b.

This is not the most accurate way of testing Coulomb's Law at laboratory scales, but it gives interesting limits. Think of what choice of a and b would increase the sensitivity to δ.

2. (a) We can assume the two daughter nuclei are point charges, with charges of $q_{Kr} = 36q_e$ and $q_{Ba} = 56q_e$. The potential energy of the two nuclei a distance r apart is given by

$$\begin{aligned} \Phi(r_{sep}) &= -\frac{k q_{Kr} q_{Ba}}{r_{sep}} & (28) \\ \Rightarrow r_{sep} &= -\frac{k q_{Kr} q_{Ba}}{\Phi(r_{sep})} & (29) \\ &= -\frac{(8.987 \times 10^9 \text{ N m}^2/\text{C}^2)(36 \times 1.602 \times 10^{-19} \text{ C})((36 \times 1.602 \times 10^{-19} \text{ C})}{(-0.85 \times 200 \text{ MeV} \times (1.602 \times 10^{-13} \text{ J/MeV})} \\ &= 1.69 \times 10^{-14} \text{ m}, & (30) \end{aligned}$$

which is of the same size as the scale of the daughter nuclei.

(b) We can generalize the formula for the potential energy using as the charges for the two daughter nuclei $f q_U$ and $(1-f) q_U$. Then the potential energy becomes

$$\begin{aligned} \Phi(r_{sep}) &= \frac{k f (1-f) q_U^2}{r_{sep}} & (31) \\ &= \frac{k q_U^2}{r_{sep}} \times \left(f - f^2\right). & (32) \end{aligned}$$

We see that the energy is a quadratic function of f, and has a maximum value. The fraction that maximizes the energy, f_{max}, is found by setting the derivative of Φ w.r.t. f to zero, or

$$
\begin{aligned}
0 &= 1 - 2f_{max} && (33)\\
f_{max} &= 1/2. && (34)
\end{aligned}
$$

Its a poorly understand fact that most of the energy generated in a nuclear reaction is in fact electromagnetic in origin.

3. (a) Suppose we have a test charge q at position $\vec{x'} = (\rho', z', \phi')$ above the plane. We know that an image charge located at $(\rho', -z', \phi')$ and charge $-q$ would result in a potential field equal to zero everywhere for $z = 0$. Although this can't be the full story, we start with this and see what can then be next done. Explicitly, the Green function is the potential function Φ_1 divided by the test charge:

$$
G_1(\vec{x}, \vec{x'}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\left|\vec{x} - \vec{x'}\right|} - \frac{1}{\left|\vec{x} - \vec{x'} + 2z'\hat{z}\right|}\right]. \qquad (35)
$$

We note that the other boundary condition is independent of the test charge and so is an additive term to the general solution to Φ using Green's Theorem. To take into account this boundary condition, namely that $\Phi = V$ in the region $z = 0$ and $\rho < a$, we consider the effects of placing two insulating disks slightly less than radius a, D and D', each having total charge Q, and $-Q$, respectively. The two disks are separated in z by a small amount, Δz with disc D located $\Delta z/2$ above the plane $z = 0$ and disk D' $\Delta z/2$ below. The potential associated with these two charge distributions at position $\vec{x}$ is

$$
\Phi_2(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{a}\left[\frac{\sigma}{\left|\vec{x} - \vec{x'}\right|} - \frac{\sigma}{\left|\vec{x} - \vec{x'} + \Delta z\hat{z}\right|}\right] d\phi'\rho' d\rho'. \qquad (36)
$$

where we have defined the surface charge density $\sigma \equiv Q/(\pi a^2)$. Since

$$
\begin{aligned}
\frac{1}{\left|\vec{x} - \vec{x'} + \Delta z\hat{z}\right|} &= \frac{1}{\sqrt{\left|\vec{x} - \vec{x'}\right|^2 + (\Delta z)^2 + 2\Delta z\left(\vec{x} - \vec{x'}\right)\cdot\hat{z}}} && (37)\\
&= \frac{1}{\left|\vec{x} - \vec{x'}\right|}\left[1 + \frac{2\Delta z\left(\vec{x} - \vec{x'}\right)\cdot\hat{z}}{\left|\vec{x} - \vec{x'}\right|^2} + O\left[(\Delta z)^2\right]\right]^{-1/2} && (38)\\
&= \frac{1}{\left|\vec{x} - \vec{x'}\right|}\left[1 - \frac{\Delta z\left(\vec{x} - \vec{x'}\right)\cdot\hat{z}}{\left|\vec{x} - \vec{x'}\right|^2} + O\left[(\Delta z)^2\right]\right], && (39)
\end{aligned}
$$

we can write

$$
\Phi_2(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{a}\left[\frac{\sigma}{\left|\vec{x} - \vec{x'}\right|}\right]\left[\frac{\cos\theta\Delta z}{\left|\vec{x} - \vec{x'}\right|}\right] d\phi'\rho' d\rho', \qquad (40)
$$

where θ is the angle between $\vec{x} - \vec{x'}$ and $\hat{z}$.

We now keep $\sigma \Delta z \equiv D_p$ constant while we take the limit $\Delta z \rightarrow 0$. In that case, the potential becomes

$$\Phi_2(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{a} \left[\frac{D_p \cos\theta}{\left|\vec{x} - \vec{x'}\right|^2} \right] d\phi' \rho' d\rho'. \tag{41}$$

As discussed in Jackson, pp. 33-34, this can be rewritten simply as

$$\Phi_2(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{a} D_p d\Omega' \tag{42}$$

where $d\Omega'$ is the solid angle subtended at $\vec{x} - \vec{x'}$ by the area element $\rho' d\rho' d\phi'$. The key point here is that on the surface of the disk, the potential associated with this "dipole layer" is given by

$$\Phi_2 = \frac{D_p}{\epsilon_0}. \tag{43}$$

So selecting $D_p = V\epsilon_0$ gives us the required boundary condition.

Notice that this term ensures that the solution satisfies the boundary conditions, and is therefore an additive term to the potential.

The Green function for these boundary conditions is formally then given by

$$G_2(\vec{x}, \vec{x'}) = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{\left|\vec{x} - \vec{x'}\right|} - \frac{1}{\left|\vec{x} - \vec{x'} + 2z'\hat{z}\right|} \right] + \frac{V}{4\pi} \int_0^{2\pi} \int_0^{a} d\Omega' \tag{44}$$

where $d\Omega'$ depends on both $\vec{x}$ and $\vec{x'}$ in the manner described above.

(b) Now that we have the expression for the Green function $G_2(\vec{x}, \vec{x'})$ in Eq. 44, for an arbitrary charge distribution $\kappa(\vec{x'})$, the potential at any point $\vec{x}$ is simply

$$\Phi(\vec{x}) = \int_{z>0} \frac{1}{4\pi\epsilon_0} \left[\frac{1}{\left|\vec{x} - \vec{x'}\right|} - \frac{1}{\left|\vec{x} - \vec{x'} + 2z'\hat{z}\right|} \right] \kappa(\vec{x'}) \rho' d\rho' dz' d\phi' + \frac{V}{4\pi} \int_0^{2\pi} \int_0^{a} d\Omega'. \tag{45}$$

(c) The potential along the $\hat{z}$ axis is just given by the second term in the Green function:

$$\Phi_2(\vec{x}) = \frac{V}{4\pi} \int_0^{2\pi} \int_0^{a} \left[\frac{\cos\theta}{\left|\vec{x} - \vec{x'}\right|^2} \right] d\phi' \rho' d\rho'. \tag{46}$$

The cosine term for a point $\vec{x}$ along $\hat{z}$ is

$$\cos\theta = \frac{z}{\sqrt{(\rho')^2 + z^2}} \tag{47}$$

so the integral becomes

$$\Phi_2(\vec{x}) = \frac{Vz}{4\pi}\int_0^{2\pi}\int_0^{a}\left[\frac{1}{\left[(\rho')^2+z^2\right]^{3/2}}\right]d\phi'\rho' d\rho' \tag{48}$$

$$= \frac{Vz}{2}\int_0^{a}\frac{\rho'}{z^3\left[1+\left(\frac{\rho'}{z'}\right)^2\right]^{3/2}}\,d\rho' \tag{49}$$

$$= V\left[1-\frac{z}{\sqrt{z^2+a^2}}\right] \tag{50}$$

where we have performed the integration by transforming to the angle $\theta' = \arctan(\rho'/z)$.