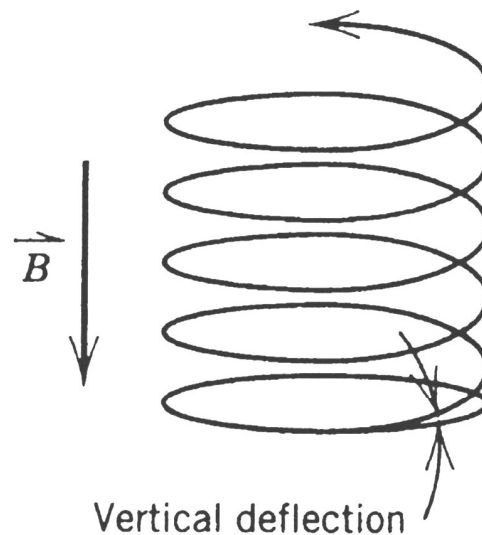
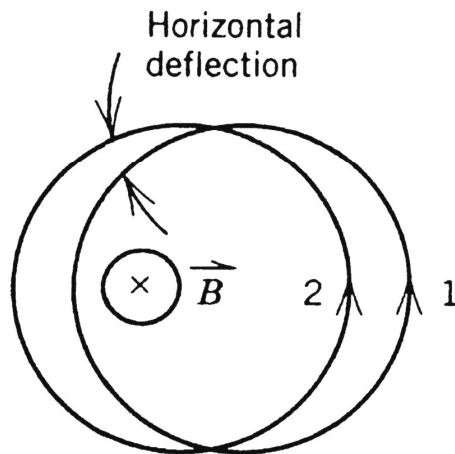


FIRST LOOK AT ORBIT STABILITY

- FIRST CYCLOTRONS AND BETA TRONS DID NOT IMMEDIATELY WORK.
- SETTING $\text{LORENTZ} = \text{CENTRIPETAL}$ ONLY WORKS FOR ONE SPECIAL ORBIT
 $\text{EQUILIBRIUM / SYNCHRONOUS ORBIT}$
- FOR MACHINE TO WORK, ONE WOULD HAVE TO INJECT EXACTLY ONTO THAT ORBIT.

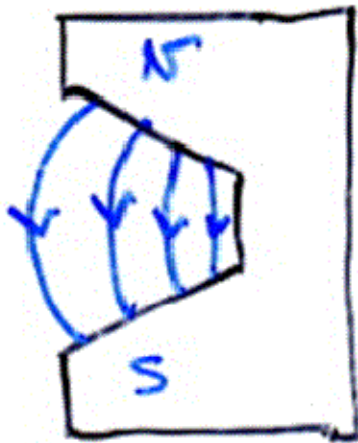


SLIGHT
DISPLACEMENT
FROM EQUILIB
ORBIT
↓
PARTICLE LOST

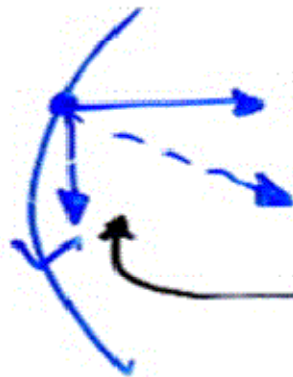
SLIGHT DISPLACEMENT FROM EQUILIBRIUM ORBIT

→ LOSS OF PARTICLES

→ NEED HORIZONTAL & VERTICAL FOCUSING



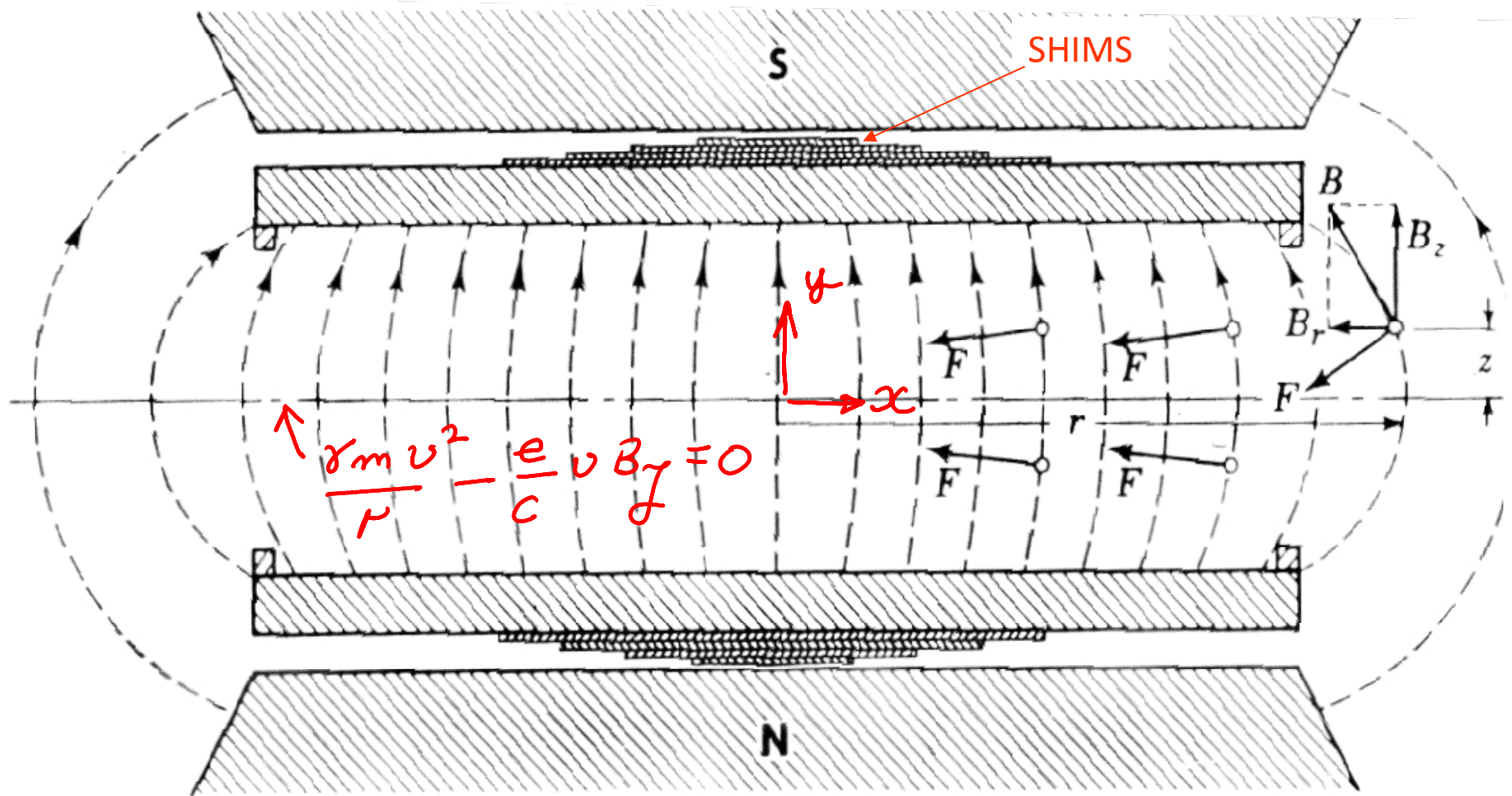
SHAPE POLE-FACES
OF GUIDE MAGNET
cf BETA TRONS



VERTICAL FOCUSING MAGNETIC FIELD

GUIDE FIELD

Orbital Stability in a Cyclotron



$$\vec{F} = \frac{q}{c} \vec{v} \times \vec{B}$$

WEAK FOCUSING

CAN WRITE A RADIALY DECREASING FIELD

$$B_y = B_{y_0} \left(\frac{R}{r} \right)^n$$

← FIELD INDEX
← RADIUS OF EQUILIBRIUM ORBIT
← VERTICAL FIELD AT R

$$\frac{\partial B_y}{\partial r} = B_{y_0} \frac{R^n}{r^n} \cdot \frac{-n}{r} \rightarrow n = \frac{-r}{B_{y_0}} \cdot \frac{r^n}{R^n} \cdot \frac{\partial B_y}{\partial r} = \frac{-r}{B_y} \frac{\partial B_y}{\partial r}$$

DEFN of B_{y_0}

FOR SMALL EXCURSIONS FROM EQUILIBRIUM ORBIT

$$r \approx R \rightarrow n = \frac{-R}{B_{y_0}} \cdot \frac{\partial B_y}{\partial r}$$

AT EQUILIBRIUM ORBIT

$$\frac{\gamma m v^2}{R} - \frac{e}{c} \cdot v \cdot B_{y_0} = 0$$

AWAY FROM EQUILIBRIUM ORBIT

RESTORING FORCE

$$F_{oc} = \frac{\gamma m v^2}{r} - \frac{e}{c} v B_y \quad (1)$$

WRITE RADIAL POSITION $r = R + \Delta r = R \left(1 + \frac{\Delta r}{R} \right)$

ORBIT RADIUS \uparrow EQUILIBRIUM ORBIT \uparrow SMALL RADIAL EXCURSION

$$B_y = B_{0y} + \frac{\partial B_y}{\partial x} \cdot \Delta r = B_{0y} \left(1 + \frac{R}{B_{0y}} \cdot \frac{\partial B_y}{\partial x} \cdot \frac{1}{R} \cdot \Delta r \right)$$

SUBST INTO $F_x = \frac{\gamma m v^2}{r} - \frac{e}{c} v B_y$

$$F_x = \frac{\gamma m v^2}{R} \left(1 + \frac{\Delta r}{R} \right)^{-1} - \frac{e}{c} v B_{0y} \left(1 + \frac{R}{B_{0y}} \cdot \frac{\partial B_y}{\partial x} \cdot \frac{\Delta r}{R} \right)$$

FOR $\Delta r \ll R \rightarrow \left(1 + \frac{\Delta r}{R} \right)^{-1} \approx \left(1 - \frac{\Delta r}{R} \right)$

AND $n = -\frac{R}{B_{0y}} \frac{\partial B_y}{\partial x}$

$$F_x = \frac{\gamma m v^2}{R} \left(1 - \frac{\Delta r}{R} \right) - \frac{e}{c} \cdot v \cdot B_{0y} \left(1 - \frac{n \Delta r}{R} \right) \quad (2)$$

$$F_x = \frac{\delta m v^2}{R} \left(1 - \frac{\alpha}{R}\right) - \frac{e}{c} \cdot v \cdot B_{0y} \left(1 - \frac{n \alpha}{R}\right) \quad (2)$$

ON EQUILIBRIUM $\frac{\delta m v}{R} = \frac{e}{c} \cdot v B_{0y}$

$$F_x = -\frac{\delta m v^2}{R} \cdot \frac{\alpha}{R} (1-n) \quad \text{RESTORING FORCE FOR } n < 1$$

$F_x \propto x$

SIMPLE HARMONIC MOTION $F_x = -kx$; $F_x = \delta m \ddot{x}$

$$\ddot{x} + \omega_x^2 \cdot x = 0; \quad \omega_x^2 = \frac{k}{\delta m}$$

IN THIS CASE HAVE SIM OSCILLATIONS FOR

$$k = \frac{v^2}{R^2} (1-n) \cdot \delta m \Rightarrow \omega_x = \frac{v}{R} \sqrt{1-n} = \omega_0 \sqrt{1-n}$$

$n < 1$ FOR STABLE OSCILLATIONS AROUND EQUILIBRIUM ORBIT

BETATRON OSCILLATIONS \rightarrow ANY CIRCULAR ACCELERATOR

THIS TREATMENT IS IN HORIZONTAL PLANE
MAIN RESTORING EFFECT FOR CORRECT
FIELD GRADIENT IS

INTERPLAY OF LORENZ & CENTRIFUGAL
IF A PARTICLE MOVES $R \rightarrow R + \alpha$

B_y DECREASES $\rightarrow F_L$ DECREASES

R INCREASES $\rightarrow F_C$ DECREASES

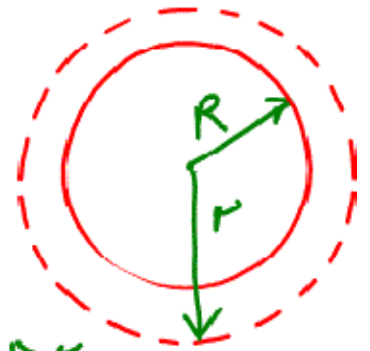
$$F_C = \frac{\gamma m v^2}{r} \quad ; \quad F_L = e B_y \cdot v$$

STABILITY DEPENDS ON WHICH OF
THESE FORCES IS GREATER, AFTER
A DISPLACEMENT FROM

THE EQUILIBRIUM ORBIT

LIMITING CASE IS WHEN ΔF_c , ΔF_L JUST GIVE EQUILIBRIUM AT LARGER RADIUS

$$\frac{\gamma_m v^2}{R} = \frac{e}{c} v B_{0y} \rightarrow \gamma_m v = \frac{e}{c} R B_{0y}$$



IF THERE IS NO CHANGE IN MOMENTUM $R \rightarrow r$ AT LARGER RADIUS

$$\gamma_m v = \frac{e}{c} \cdot r B_y = \frac{e}{c} R B_{0y} \rightarrow B_y = B_{0y} \frac{R}{r}$$

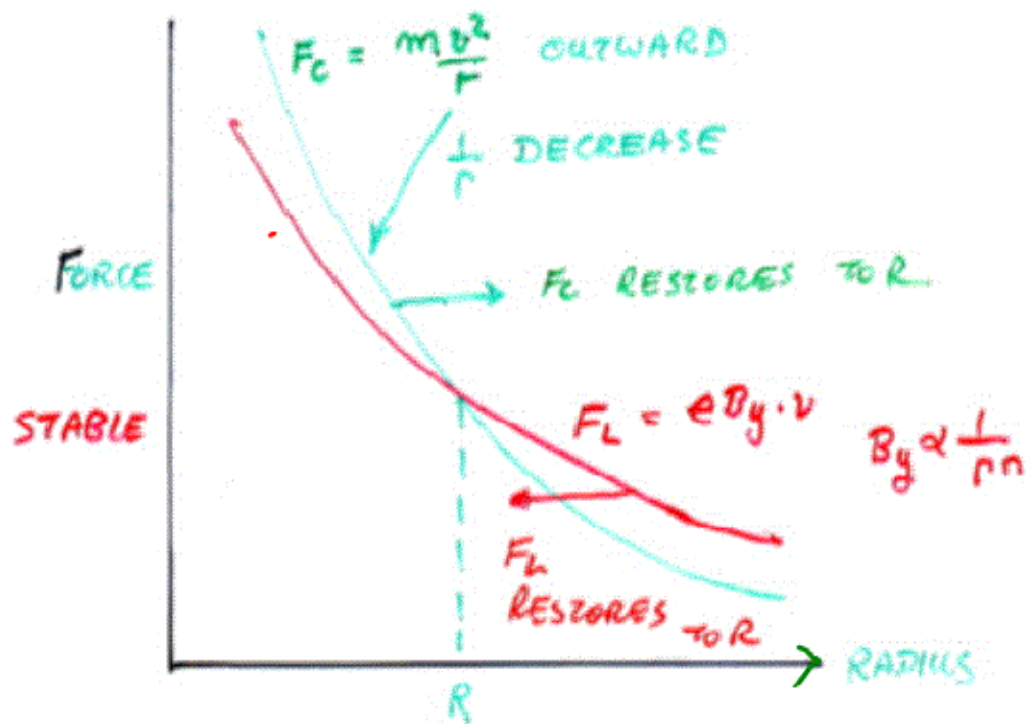
COMPARE THIS TO $B_y = B_{0y} \left(\frac{R}{r}\right)^n$

THIS SITUATION JUST CORRESPONDS TO THE LIMITING CASE OF $n=1$

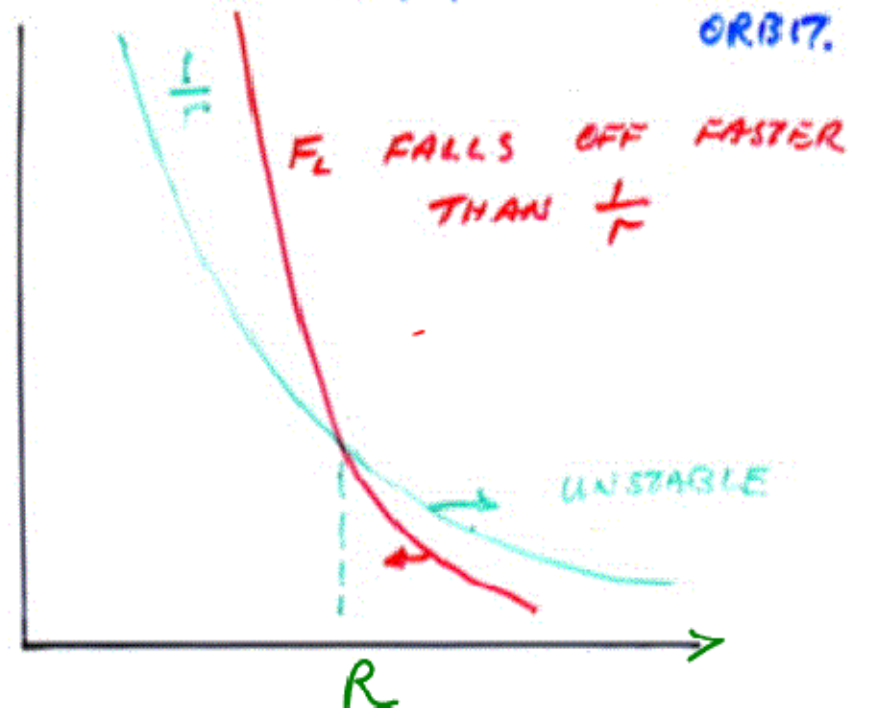
OBVIOUS FROM $\omega_{jc} = \frac{v}{R} \sqrt{1-n}$

$$B_y = B_{0y} \left(\frac{R}{r} \right)^n$$

$n < 1$ STABLE



$n > 1 \rightarrow$ UNSTABLE ORBIT.



VERTICAL BETATRON OSCILLATIONS

$$\gamma m \ddot{y} = \frac{e}{c} v B_x \quad \text{NO CENTRIFUGAL FORCE}$$

$$\vec{V} \times \vec{B} = 0 \rightarrow \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0 \rightarrow B_x = \int \frac{\partial B_x}{\partial y} dy$$

$$\text{USE } \eta = -\frac{R_0}{B_{y0}} \cdot \frac{\partial B_y}{\partial x} \rightarrow -\eta \int \frac{B_{0y}}{R} dy = -\eta \frac{B_{0y}}{R} y$$

$$\gamma m \ddot{y} = -\eta \frac{B_{0y}}{R} \cdot \frac{e}{c} \cdot v \cdot y \quad \text{USE } \frac{\gamma m v^2}{R} = \frac{e}{c} \cdot v B_{0y}$$

$$\gamma m \ddot{y} + \eta \frac{\gamma m v^2}{R^2} = 0 \rightarrow \ddot{y} + \eta \frac{v^2}{R^2} \cdot y = 0$$

SHM FOR $\eta > 0$
VERTICAL BETATRON
OSCILLATIONS

$$\omega^2 = \eta \frac{v^2}{R^2} = \eta \omega_0^2$$

$$\omega = \omega_0 \sqrt{\eta}$$

Betatron Oscillations

Horizontal

Vertical

$$B_z = B_{z_0} \left(\frac{R}{r} \right)^n$$

Field Index n

Equilibrium Orbit R

$$\gamma m \frac{d^2 y}{dt^2} = \frac{e}{c} v B_x$$

$$\frac{\gamma m v^2}{R} - \frac{e}{c} v B_{z_0} = 0$$

Centrifugal = Lorentz
on equilibrium orbit

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = 0$$

$$F_x = \frac{\gamma m v^2}{r} - \frac{e}{c} v B_z$$

Restoring Force

$$\gamma m \frac{d^2 y}{dt^2} = -n \frac{B_z}{R} \frac{e}{c} v = F_z$$

$$F_x = -\frac{\gamma m v^2}{R} \frac{x}{R} (1-n)$$

Simple Harmonic

$$\gamma m \frac{d^2 y}{dt^2} + n \gamma m \frac{v^2}{R^2} = 0$$

$$\omega_x = \frac{v}{R} \sqrt{1-n}$$

Stable Oscillations around
Equilibrium orbit

$$\omega_z = n \frac{v^2}{R^2}$$

Weak Focusing

$$1 > n > 0$$

WEAK FOCUSING

- CANNOT MAKE FIELD INDEX n ARBITRARILY LARGE IN BOTH HORIZONTAL & VERTICAL
- CANNOT MAKE RESTORING FORCES ARBITRARILY LARGE IN BOTH DIRECTIONS
- CANNOT MAKE AMPLITUDE OF HORIZONTAL AND VERTICAL BETATRON OSCILLATIONS ARBITRARILY SMALL

LARGE CROSS SECTION OF VACUUM PIPE \rightarrow LARGE MAGNETS \rightarrow \$\$\$
\$\$\$

MUST BE A BETTER WAY

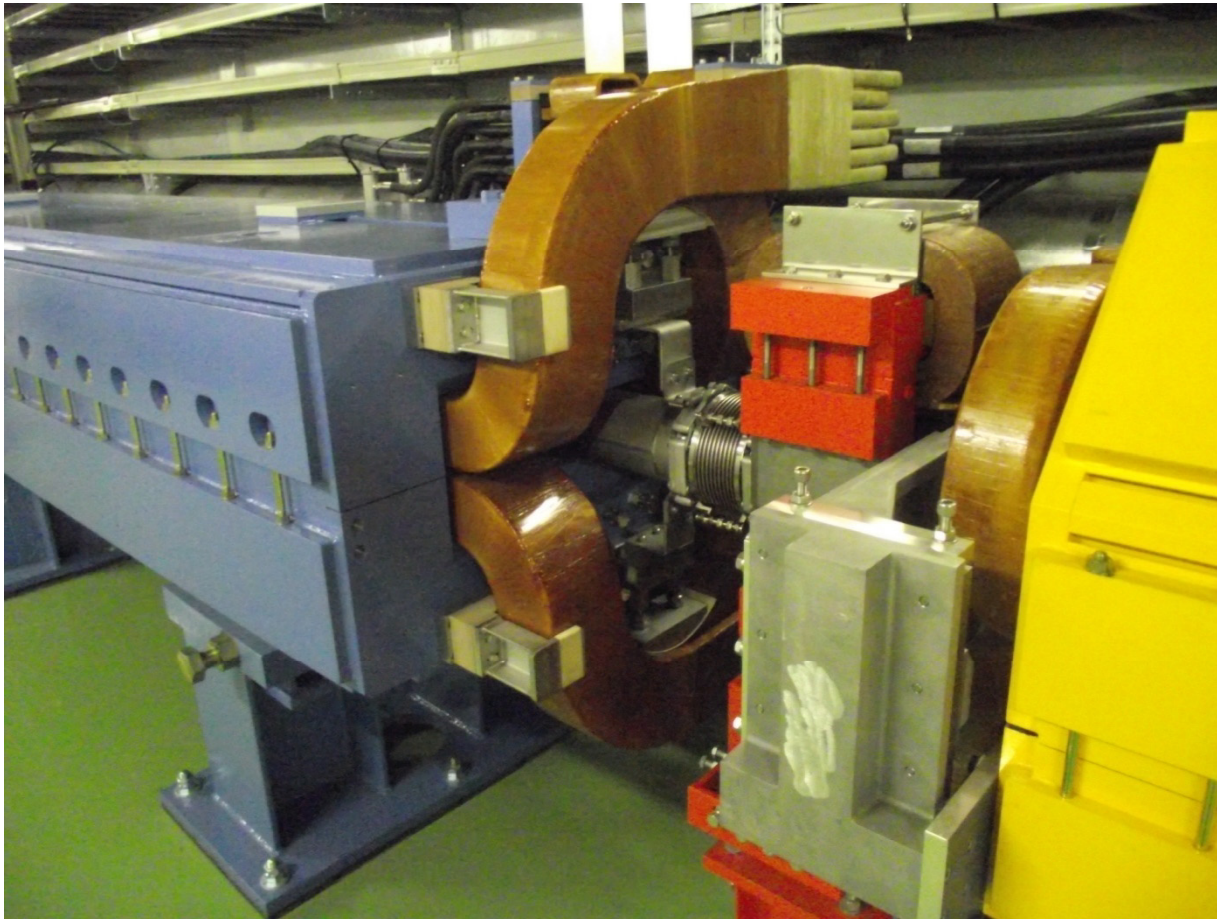
\rightarrow SHORT DIGRESSION INTO PARTICLE DYNAMICS

DYNAMICS OF PARTICLES IN MAGNETIC FIELDS

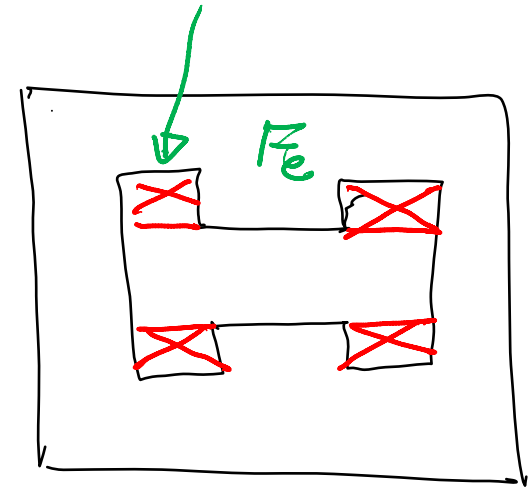
MAGNETS OF MANY MULTIPOLES USE IN
ACCELERATOR

DIPOLE — BENDING MAGNET
QUADRUPOLE — FOCUSING
6-POLE } REMOVE CHROMATIC
12-POLE } ABERATIONS

DIPOLE - BEND



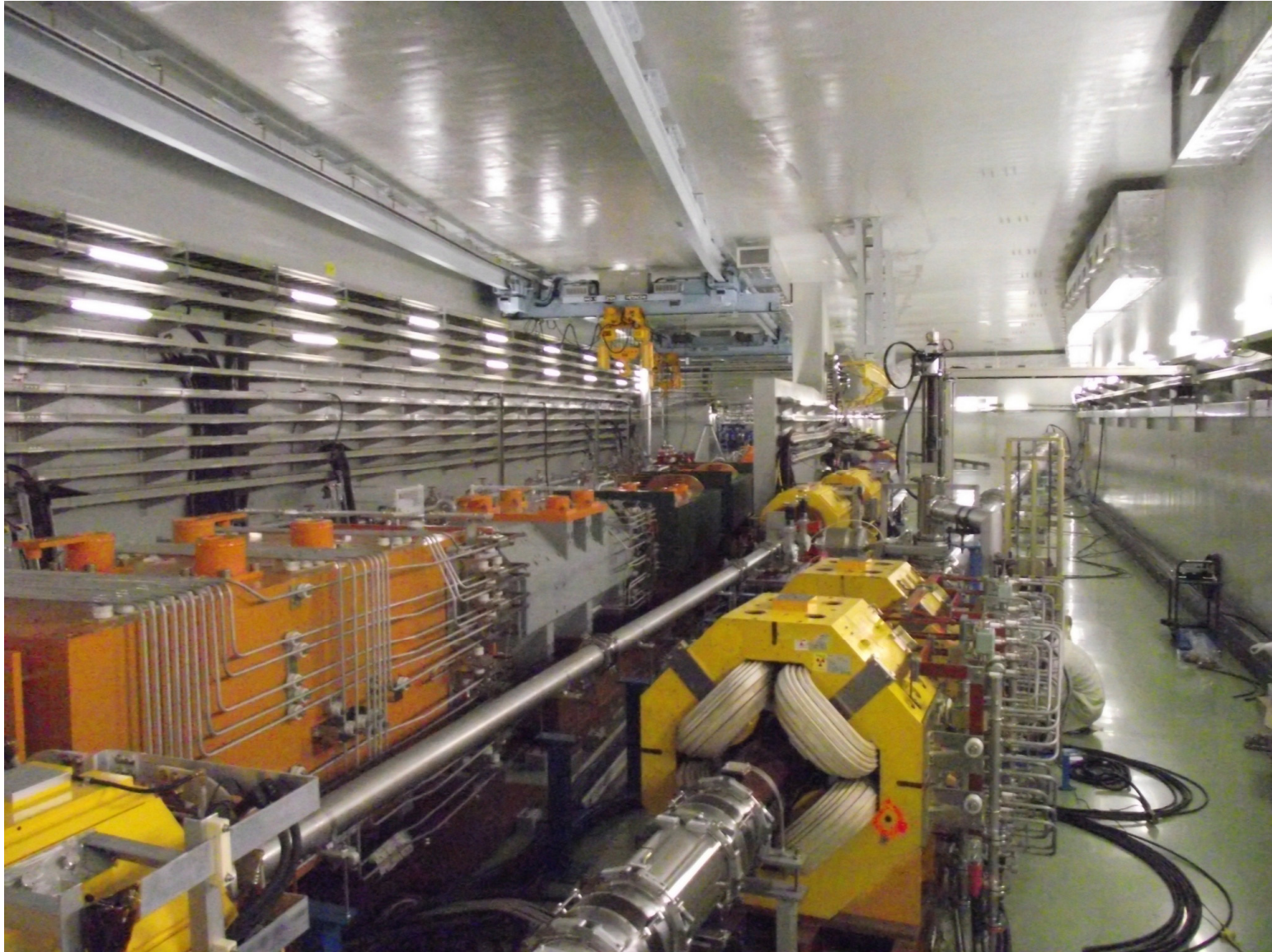
Cu COILS



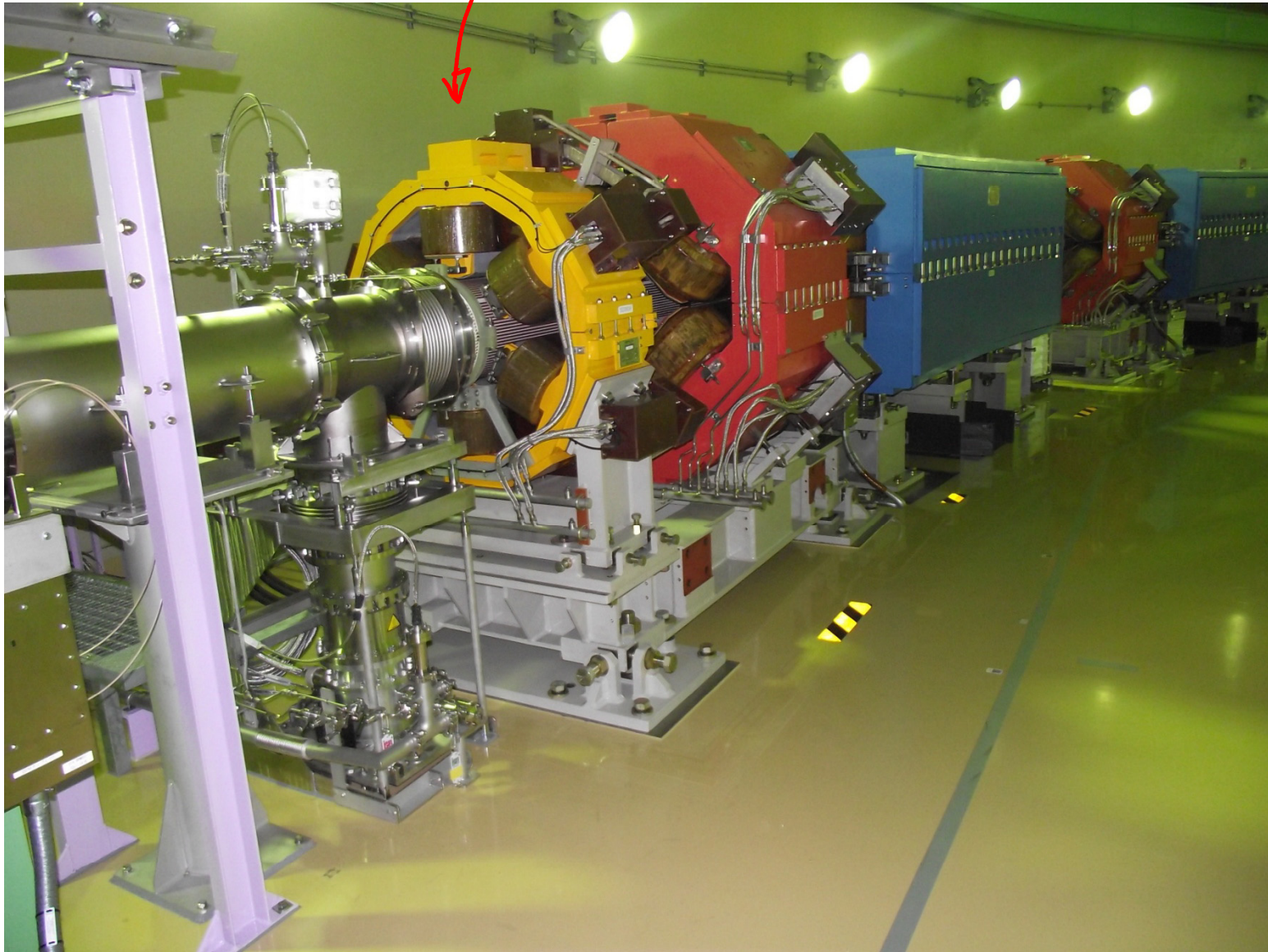
$$B = \frac{2\mu_0 NI}{h}$$

GAP BETWEEN
POLES

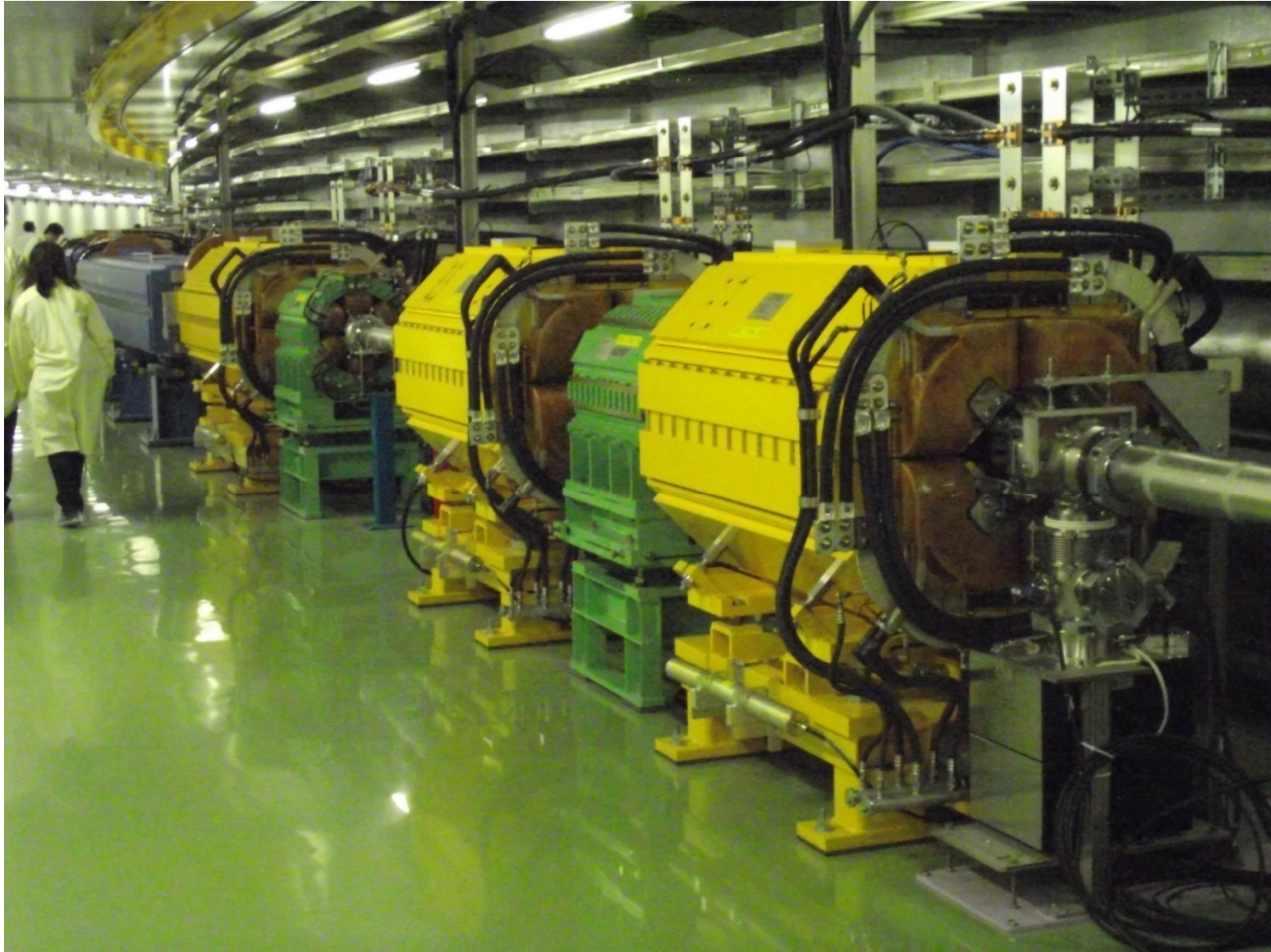
QUADRUPOLES — FOCUSING



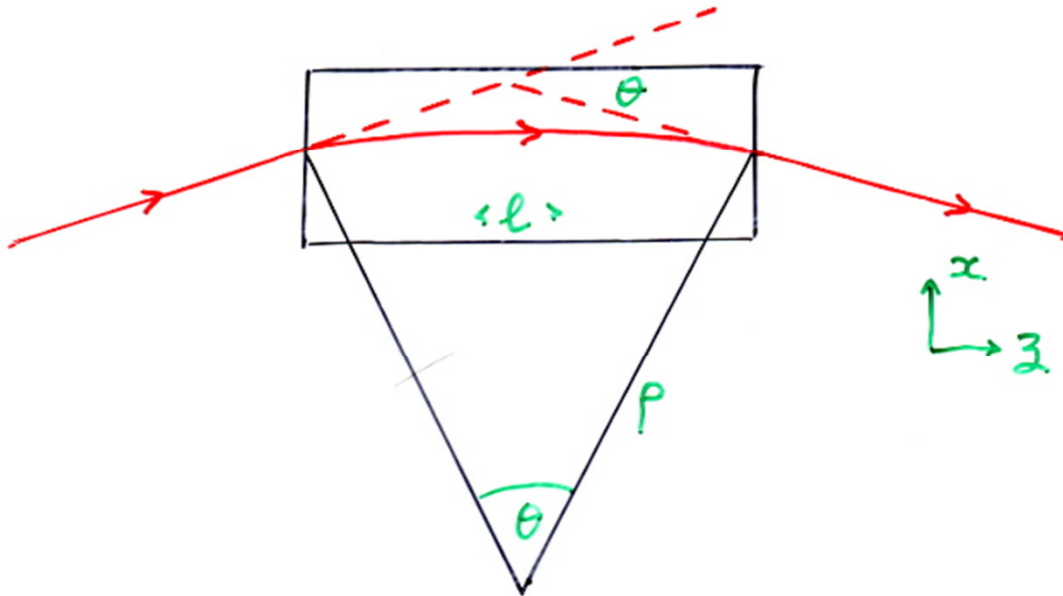
6-POLE — CORRECTION



DIPOLE \rightarrow QUAD \rightarrow 6 \rightarrow QUAD \rightarrow 6 \rightarrow QUAD



BENDING MAGNET



$$\frac{1}{\rho} = \frac{eB}{p}$$

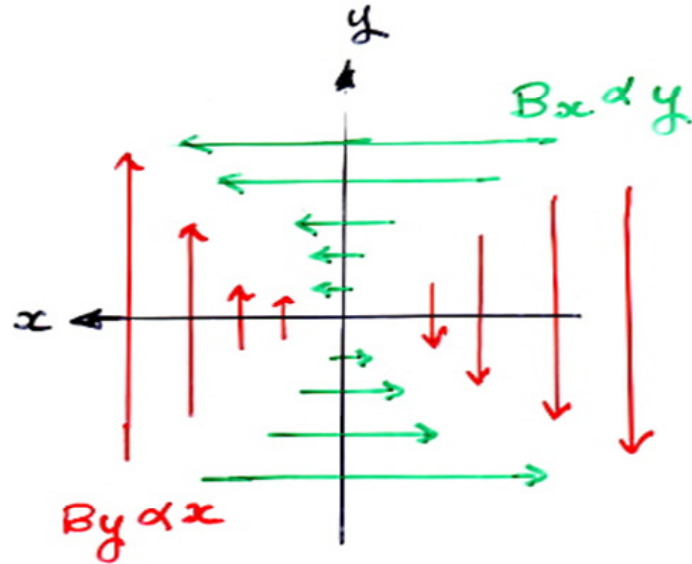
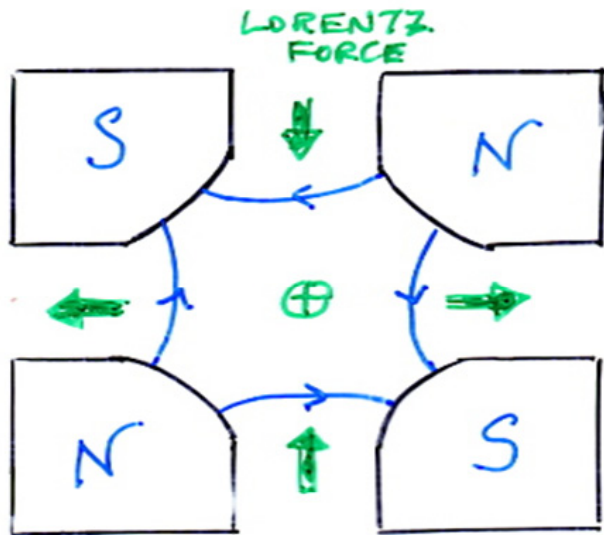
PASSING THRU DIPOLE, CHANGE IN ANGLE $\Delta\alpha' = \frac{d\alpha}{dz}$

$$\Delta\alpha' = \frac{e}{p} = \frac{eB_y}{p}$$

BUT $B_y = \frac{\partial B_y}{\partial x} \cdot x = -kx$ FIELD GRADIENT

$$\Delta\alpha' = -\frac{e}{p} \cdot k \cdot x = \theta$$

QUADRUPOLE



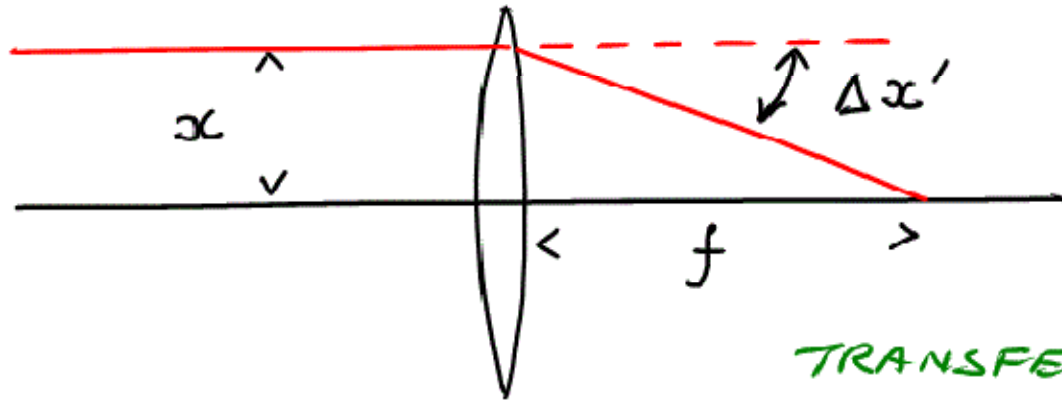
POLE FACES — EQUIPOTENTIAL SURFACE
FOR HYPERBOLIC POLE FACES

MAGNETIC SCALAR
POTENTIAL

$$\phi = k \cdot x \cdot y$$

$$\vec{B} = -\vec{\nabla}\phi \rightarrow \begin{cases} B_x = -ky & \text{FOCUS} \\ B_y = -kx & \text{DEFOCUS} \end{cases}$$

WARM IRON $k \sim 12.7 \text{m}^{-1}$ FOR 10 GeV e^-
SUPERCONDUCTING $k \sim 75 \text{Tm}^{-1}$ FOR 1 TeV p



DEFINITION OF FOCAL LENGTH

$$\Delta x' = -\frac{x}{f}$$

$$\frac{1}{f} = -\frac{\Delta x'}{x} = \frac{e \cdot k \cdot l}{p}$$

FIELD GRADIENT
OF QUADRUPOLE

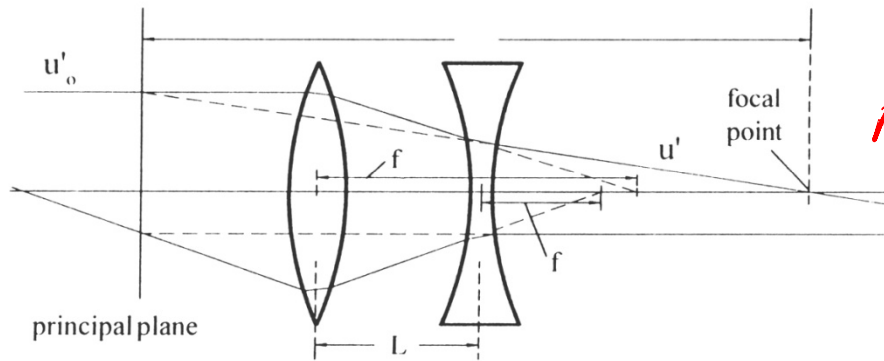
TRANSFER MATRIX

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{OUT}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{IN}}$$

THIN LENS \rightarrow

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_L$$

FIELD FREE REGION

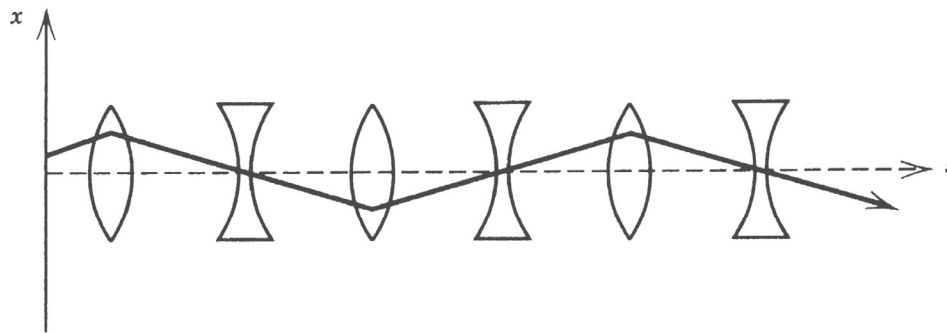


CAN HAVE STRONG FOCUSING IN BOTH PLANES,

Fig. 4.14. Focusing in a quadrupole doublet

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{L}{f} & L \\ -\frac{L}{f^2} & 1 - \frac{L}{f} \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 0 \\ -\frac{L}{f^2} & 1 \end{pmatrix} \quad L \ll f \quad \text{Net Focusing}$$



- FODO Lattice
- Strong Focusing

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & L \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 - \frac{L}{f_2} & L \\ \frac{1}{f_1} - \frac{1}{f_2} - \frac{L}{f_1 f_2} & 1 + \frac{L}{f_1} \end{pmatrix}$$

$$f_1 = f_2 \rightarrow -\frac{L}{f_2}$$

CONDITION FOR FOCUSING

THIN LENS

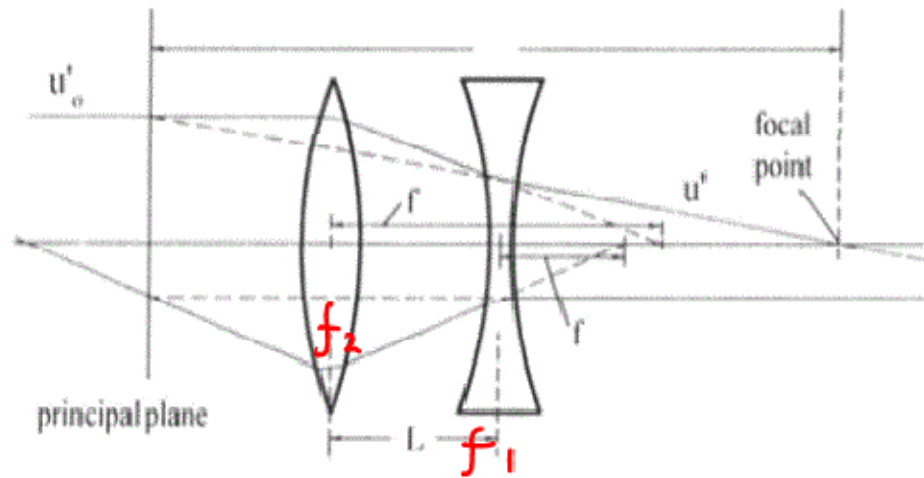


Fig. 4.14. Focusing in a quadrupole doublet

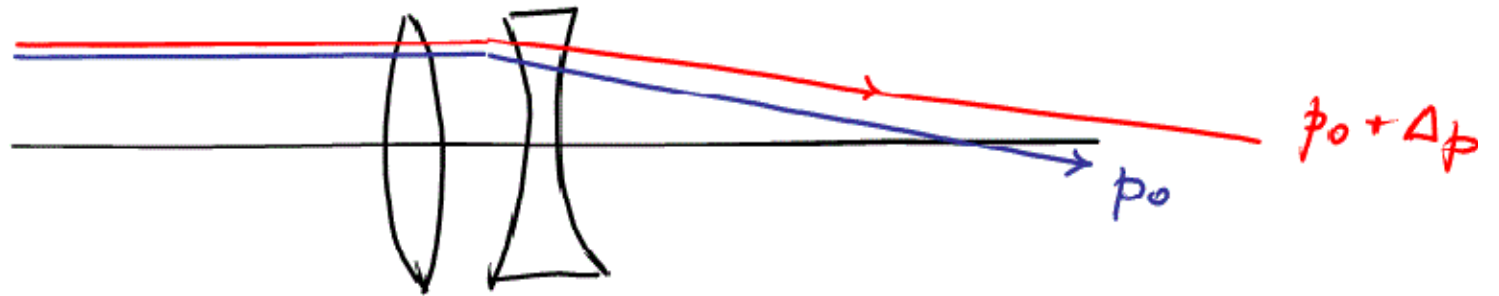
FINITE LENGTH

$$\begin{pmatrix} \cos \phi & \sin \phi / \sqrt{k} \\ -\sqrt{k} \sin \phi & \cos \phi \end{pmatrix}_F$$

$$\begin{pmatrix} \cosh \phi & \sinh \phi / \sqrt{k} \\ \sqrt{k} \sinh \phi & \cosh \phi \end{pmatrix}_D$$

$$\phi = \sqrt{k} \cdot L$$

NON EQUILIBRIUM ORBIT



IN REAL LIFE HAVE FINITE RANGE OF MOMENTA

→ CHROMATIC ABERRATION

→ CORRECT WITH SEXTAPOLES

STABILITY OF NONEQUILIBRIUM ORBITS IN FODO LATTICE

- PARTICLE TRAVERSES A SERIES OF ELEMENTS
TRANSFER MATRICES $M = M_1 M_2 \dots M_n$
- PERIODIC STRUCTURE — I.E. TRAVERSE
SEQUENCE REPETITIVELY
- AFTER n TRAVERSALS, TRANSFER MATRIX

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

FOR STABILITY, THIS
MUST BE FINITE FOR
LARGE n

$M \rightarrow V_1 \quad V_2$ EIGEN VECTOR

$\lambda_1 \quad \lambda_2$ EIGEN VALUES

ANY
INITIAL

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{in} = A \overbrace{V_1}^{\text{CONST}} + B \overbrace{V_2}^{\text{CONST}}$$

n TIMES THROUGH
PERIODIC STRUCTURE
 \downarrow

$$M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{in} = A \lambda_1^n V_1 + B \lambda_2^n V_2$$

• STABILITY REQUIRES λ_1^n, λ_2^n NOT GROW WITH n

$$\det(M_i) = 1 \rightarrow \det(M) = 1 \rightarrow \lambda_2 = 1/\lambda_1$$

WRITE $\left. \begin{array}{l} \lambda_1 = e^{i\mu} \\ \lambda_2 = e^{-i\mu} \end{array} \right\} \mu \text{ IS REAL FOR STABLE OSCILLATIONS}$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ THEN CHARACTERISTIC EQUATION $\det(M - \lambda I) = 0$

BECOMES
$$\underbrace{(ad - bc)}_{\det M = 1} - (a + d)\lambda + \lambda^2 = 0$$

$$\underbrace{\lambda^{-1} + \lambda}_{e^{i\mu} + e^{-i\mu}} = a + d = \text{Tr } M \quad \text{REAL}$$

$$e^{i\mu} + e^{-i\mu} = 2 \cos \mu = \text{Tr } M$$

STABILITY FOR $-1 \leq \frac{1}{2} \text{Tr } M \leq 1$

FOR FODO WITH EQUALLY SPACED MAGNETS

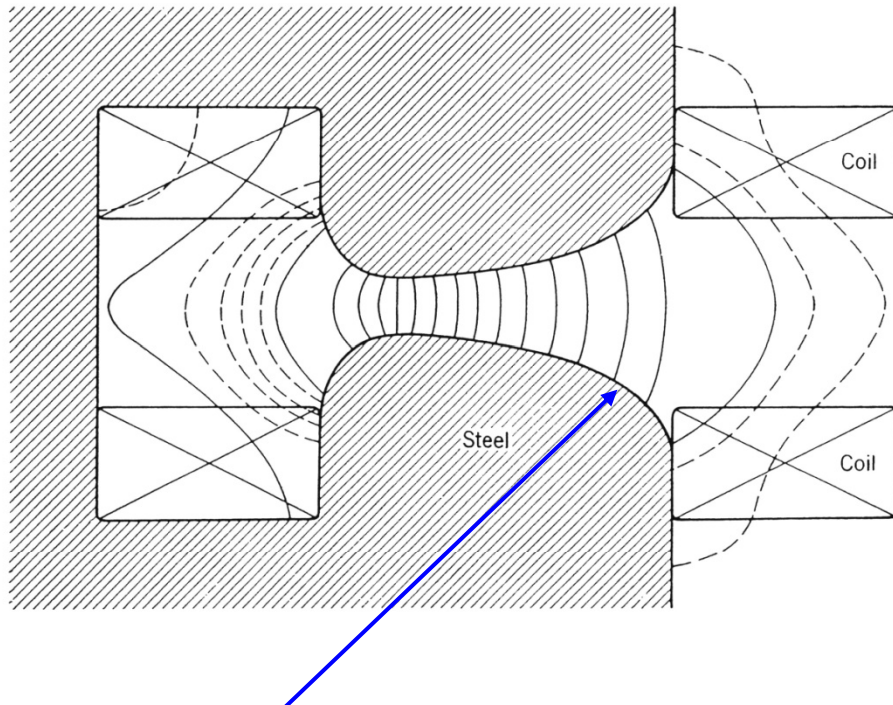


$$M = \begin{pmatrix} 1 + L/f & 2L + L^2/f \\ -L/f^2 & 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 \end{pmatrix}$$

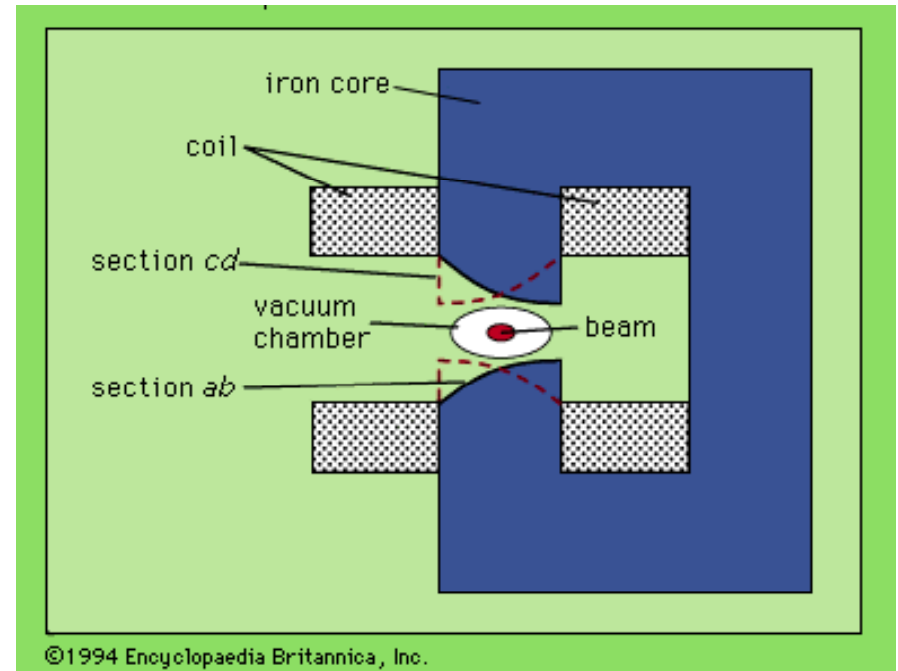
STABILITY $-1 \leq \frac{1}{2} \text{Tr} M \leq 1 \rightarrow -1 \leq \frac{1}{2} \left(2 - \left(\frac{L}{f}\right)^2 \right) \leq 1$

$\rightarrow \left| \frac{L}{2f} \right| \leq 1$ FOCAL LENGTH $> \frac{1}{2}$ ELEMENT SPACING

Strong Focusing

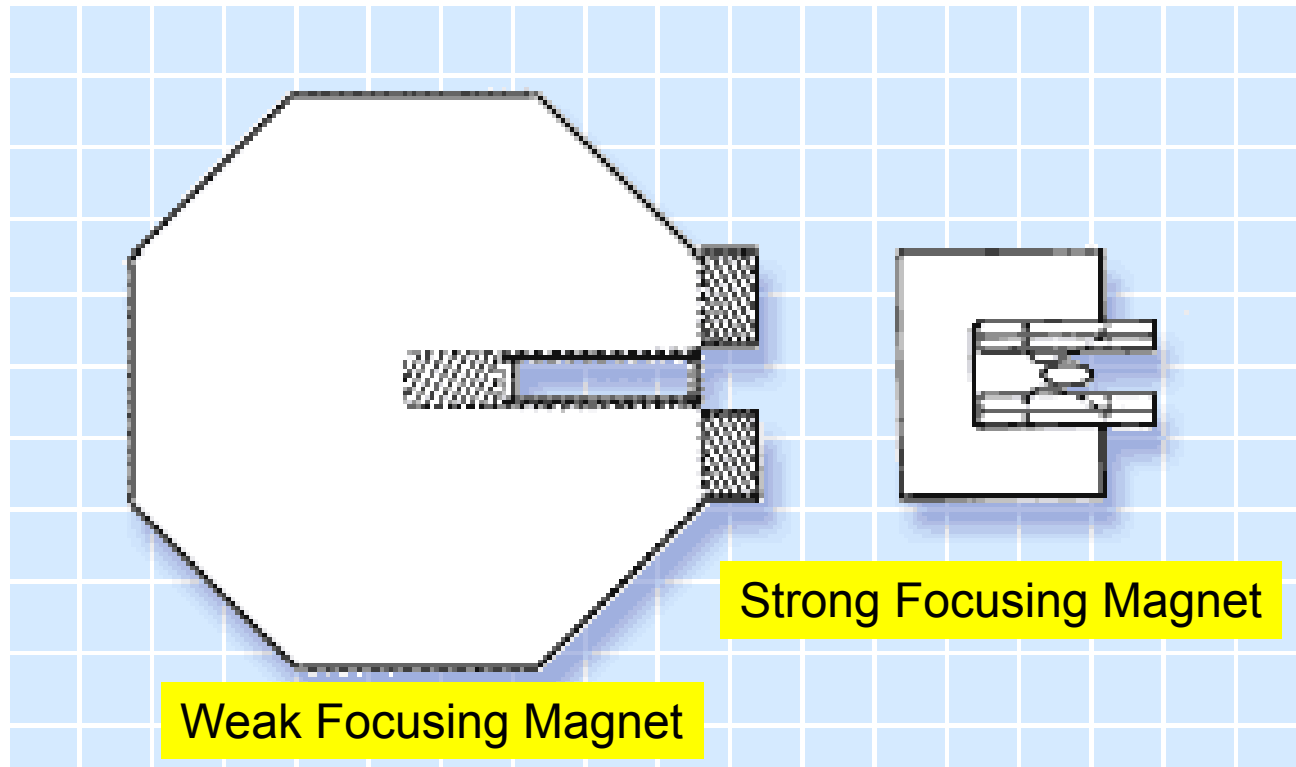


- Field Index set by Pole Face Shape
- Weak, $n = 0.5$
- Strong, $n = 3500$



- Strong Focusing = Alternating Gradient
- “Combined Function” Magnet

Enormous Cost Saving



- Strong Focusing = Alternating Gradient
- Reduce amplitude of betatron oscillations
- Reduce diameter of vacuum pipe
- Reduce Aperture of Magnets
- 35 GeV (CERN PS, AGS) costs same as 7 GeV (NIMROD)