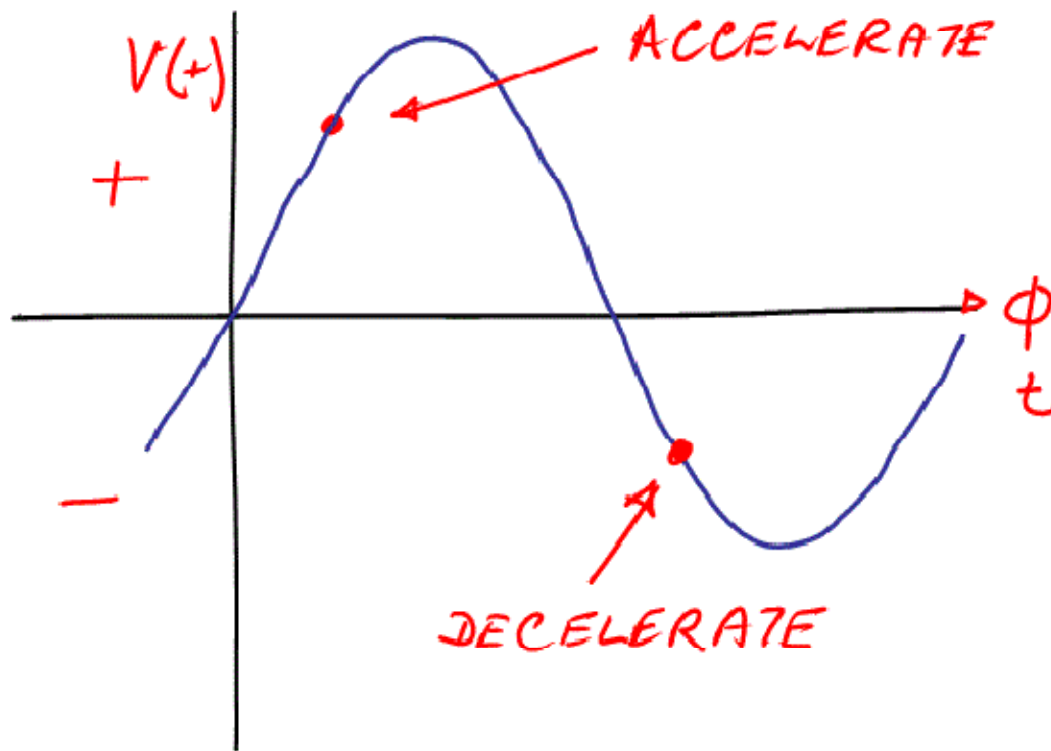


PHASE STABILITY

HAVE DISCUSSED STABILITY OF LATERAL MOTION AROUND EQUILIBRIUM ORBIT.

DOES AN NON-IDEAL ORBIT PARTICLE GET CONTINUALLY ACCELERATED \rightarrow LOOK BACK AT WIDEROE LINAC.



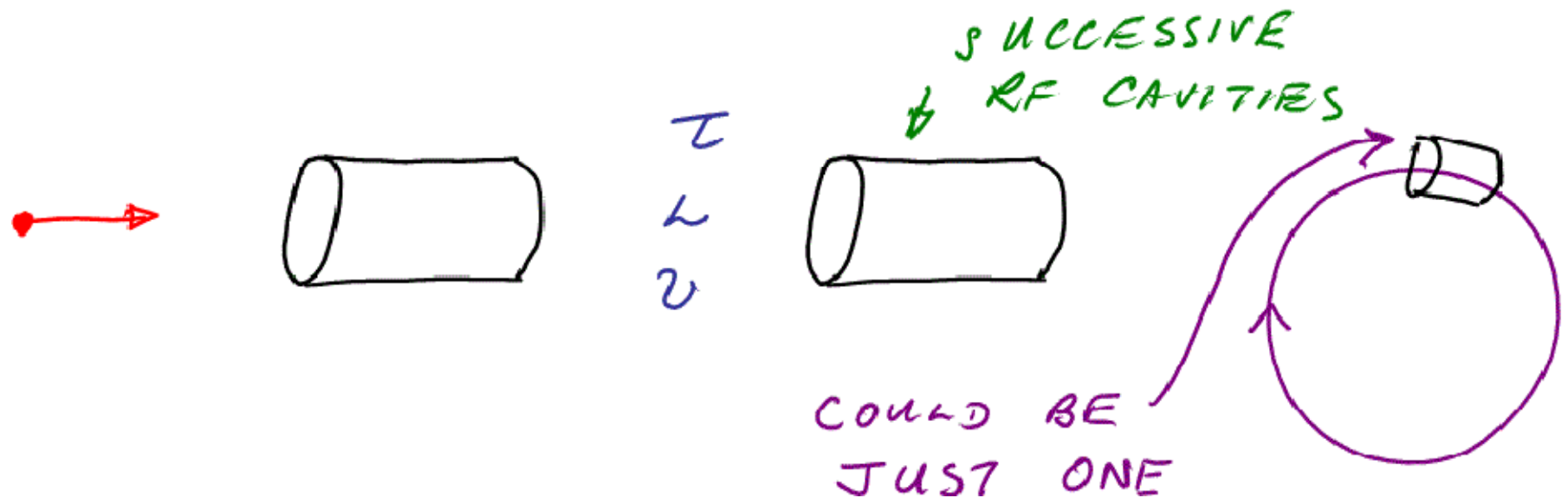
DO PARTICLES CLOSE TO IDEAL REMAIN CLOSE IN PHASE AND ENERGY

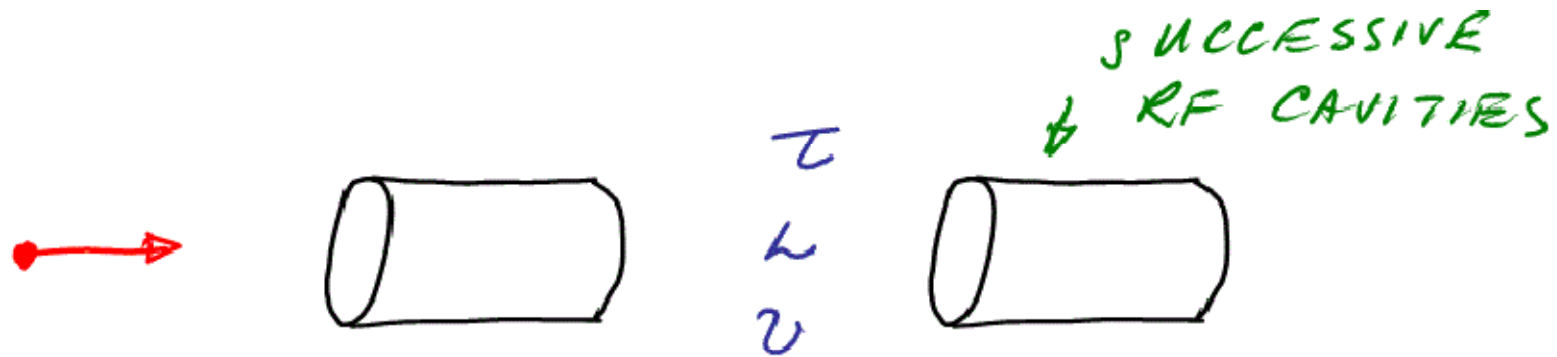
LONGITUDINAL OSCILLATIONS AROUND IDEAL PHASE

SYNCHROTRON OSCILLATIONS

SOPHISTICATED WAY TO SOLVE THIS IS TO
CONSTRUCT A HAMILTONIAN FOR THE MOTIO
→ TACKLE IN HEURISTIC FASHION

- RF CAVITY EXCITED AT FREQUENCY ω
- BY DEFINITION IDEAL PARTICLE ARRIVES
AT SAME PHASE ON EACH PASS THROUGH
THE CAVITY $\phi, (\phi + 2\pi), (\phi + 4\pi), \dots$
- RECEIVES SAME ENERGY INCREMENT ON
EACH PASS





τ TIME FOR IDEAL PARTICLE TO PASS
SUCCESSIVE CAVITIES

L DISTANCE BETWEEN CAVITIES

$$\tau = \frac{L}{v} \quad \text{"DIFFERENTIATE" AS A QUOTIENT}$$

FRACTIONAL CHANGE IN τ FOR
NON-IDEAL PARTICLE

$$\frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

$$\frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

$v > v_{\text{IDEAL}}$ LESS TIME FOR TRANSIT
 $L > L_{\text{IDEAL}}$ MORE TIME TO TRANSIT

HAVE $\phi = \gamma m v \rightarrow \frac{\Delta v}{v} = \frac{1}{\gamma^2} \left(\frac{\Delta \phi}{\phi} \right)$

NOW $\Delta L/L$ COULD ALSO DEPEND ON ϕ

IN A CIRCULAR ACCELERATOR $\phi \uparrow \rightarrow R \uparrow \rightarrow L \uparrow$

SO DEFINE, BY ANALOGY

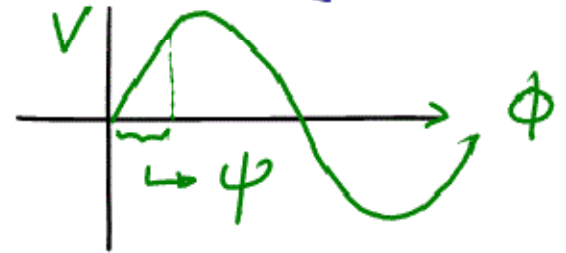
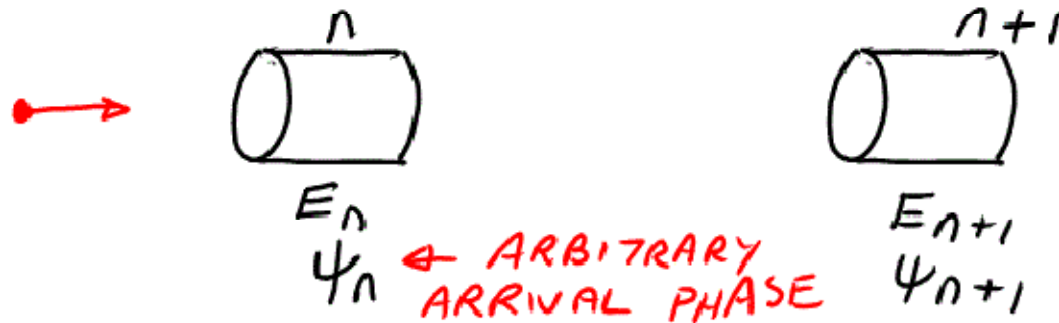
$$\frac{\Delta L}{L} = \frac{1}{\gamma_t^2} \left(\frac{\Delta \phi}{\phi} \right)$$


 PARAMETER WHICH DEPENDS ON THE GEOMETRY OF THE ACCELERATOR

$$\frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v} \rightarrow \frac{\Delta \tau}{\tau} = \eta \cdot \frac{\Delta \phi}{\Delta p} = \left(\frac{1}{\gamma_E^2} - \frac{1}{\gamma^2} \right) \frac{\Delta \phi}{\phi}$$

η - SLIP FACTOR

THINK ABOUT ENERGY CHANGE - THIS COULD BE SUCCESSIVE KICKS IN CIRCULAR ACCEL.



FREQUENCY IDEAL

PHASES RELATED BY $\psi_{n+1} = \psi_n + \omega (\tau + \Delta \tau)_{n+1}$

$$\psi_{n+1} = \psi_n + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau} \right)_{n+1}$$

NON IDEAL

$$\textcircled{1} \Psi_{n+1} = \Psi_n + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau} \right)_{n+1}$$

IDEAL PARTICLE ALWAYS ARRIVE "SAME TIME"
 REFER ALL PHASES TO CONSTANT "SAME PHASE"
 PHASE OF IDEAL PARTICLE

$$\Psi_{\text{IDEAL}} = \omega T_n \leftarrow \text{ARRIVAL TIME AT CAVITY } n$$

FOR ARBITRARY PHASE, SUBTRACT OFF "IDEAL PHASE"

$$\Phi_n = \Psi_n - \omega T_n$$

\textcircled{1} BECOMES

$$\underbrace{\Phi_{n+1} + \omega T_{n+1}}_{\Psi_{n+1}} = \underbrace{\Phi_n + \omega T_n}_{\Psi_n} + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau} \right)_{n+1}$$

FOR IDEAL PARTICLE

$$\phi_{n+1} + \omega T_{n+1} = \phi_n + \omega T_n + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta z}{z} \right)_{n+1}$$

$$T_n + \tau_{n+1} = T_{n+1}$$

$$\phi_{n+1} + \cancel{\omega T_n} + \omega \cancel{\tau_{n+1}} = \phi_n + \cancel{\omega T_n} + \omega \cancel{\tau_{n+1}} + \omega \tau_{n+1} \left(\frac{\Delta z}{z} \right)_{n+1}$$

$$\phi_{n+1} = \phi_n + \omega \tau_{n+1} \left(\frac{\Delta z}{z} \right)_{n+1}$$

IN SUCCESSIVE CAVITIES, THIS TELLS YOU
 HOW PHASE OF **NON-IDEAL PARTICLE**
 VARIE W. R. T TO PHASE OF **IDEAL**

$$\phi_{n+1} = \phi_n + \omega \tau_{n+1} \left(\frac{\Delta z}{\tau} \right)_{n+1}$$

$$\frac{\Delta z}{\tau} = \underbrace{\left(\frac{1}{\gamma^2} - \frac{1}{\gamma'^2} \right)}_{\eta} \frac{\Delta p}{p}$$

$$\tau_n \approx \tau_{n+1}$$

DIFFERENCE
EQUATION OF MOTION

$$\phi_{n+1} = \phi_n + \eta \tau \omega \left(\frac{\Delta p}{p} \right)_{n+1}$$

ASSUME $\omega \tau$ INDEPENDENT OF n
FOR A CIRCULAR ACCELERATOR

$$\omega \tau = h \cdot 2\pi$$

RF

HARMONIC NUMBER

THAT WAS PHASE \rightarrow WHAT ABOUT ENERGY

$$(E_s)_{n+1} = (E_s)_n + eV \sin \phi_s \quad (1)$$

IDEAL SYNCHRONOUS PARTICLE \nearrow \uparrow VOLTAGE \nwarrow IDEAL (SYNCHRONOUS) PHASE

FOR NON-IDEAL PARTICLE

$$E_{n+1} = E_n + eV \sin \phi_n \quad (2)$$

DIFFERENCE IN ENERGY W.R.T IDEAL $\Delta E = E - E_s$

$$(2) - (1) \rightarrow \Delta E_{n+1} = \Delta E_n + eV (\sin \phi_n - \sin \phi_s)$$

$$\text{IN } \phi_{n+1} = \phi_n + \eta \tau \omega \left(\frac{\Delta p}{p} \right)_{n+1} \quad \frac{\Delta p}{p} \rightarrow \frac{c^2}{v^2} \frac{\Delta E}{E}$$

$$\phi_{n+1} = \phi_n + \frac{\omega \tau \eta c^2}{v^2 E_s} \cdot \Delta E_{n+1}$$

DIFFERENCE EQUATION

$$\phi_{n+1} = \phi_n + \frac{\omega \tau \eta}{v^2 E_s} \Delta E_{n+1}$$

$$\Delta E_{n+1} = \Delta E_n + eV (\sin \phi_n - \sin \phi_s)$$

IF PHASE & ENERGY CONTINUOUS VARIABLES

$$\frac{d\phi}{dn} = \frac{\eta \omega \tau c^2}{v^2 E_s} \Delta E$$

$$\frac{d\Delta E}{dn} = eV (\sin \phi - \sin \phi_s)$$



TURN NUMBER

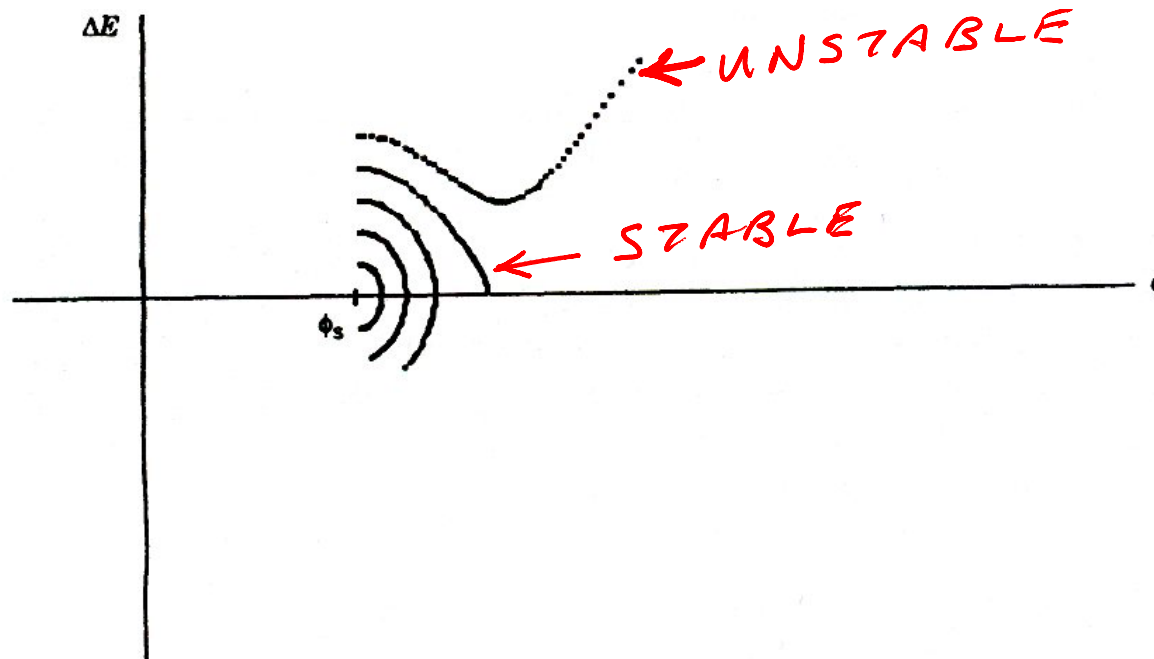


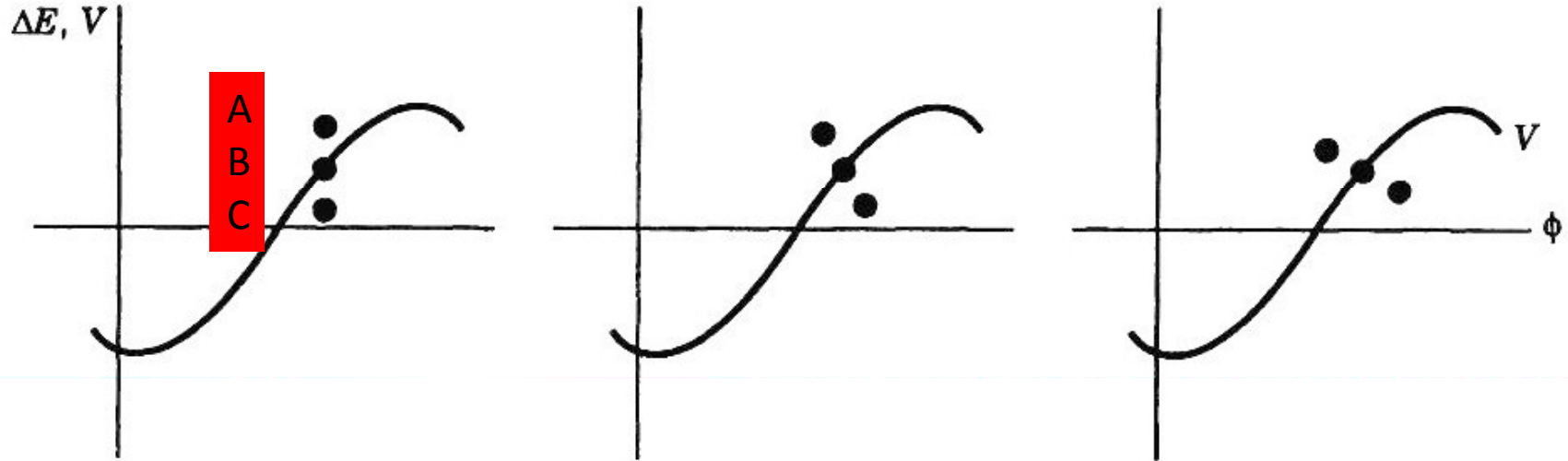
Figure 2.15. Application of the difference equations for synchrotron motion for five initial conditions. In each case, the starting value of the phase is equal to the synchronous phase.

ITERATION OF DIFFERENCE EQUATIONS OF MOTION

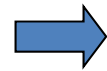
→ START FROM DIFFERENT ΔE

- UNSTABLE PARTICLE DRIFTS AWAY FROM SYNCHRONOUS PHASE
- STABLE PARTICLE ORBITS IN PHASE SPACE AROUND SYNCHRONOUS PHASE

Successive turns around accelerator lattice



- B is synchronous with RF phase
- A too energetic to be in phase
- C not energetic enough to be in phase

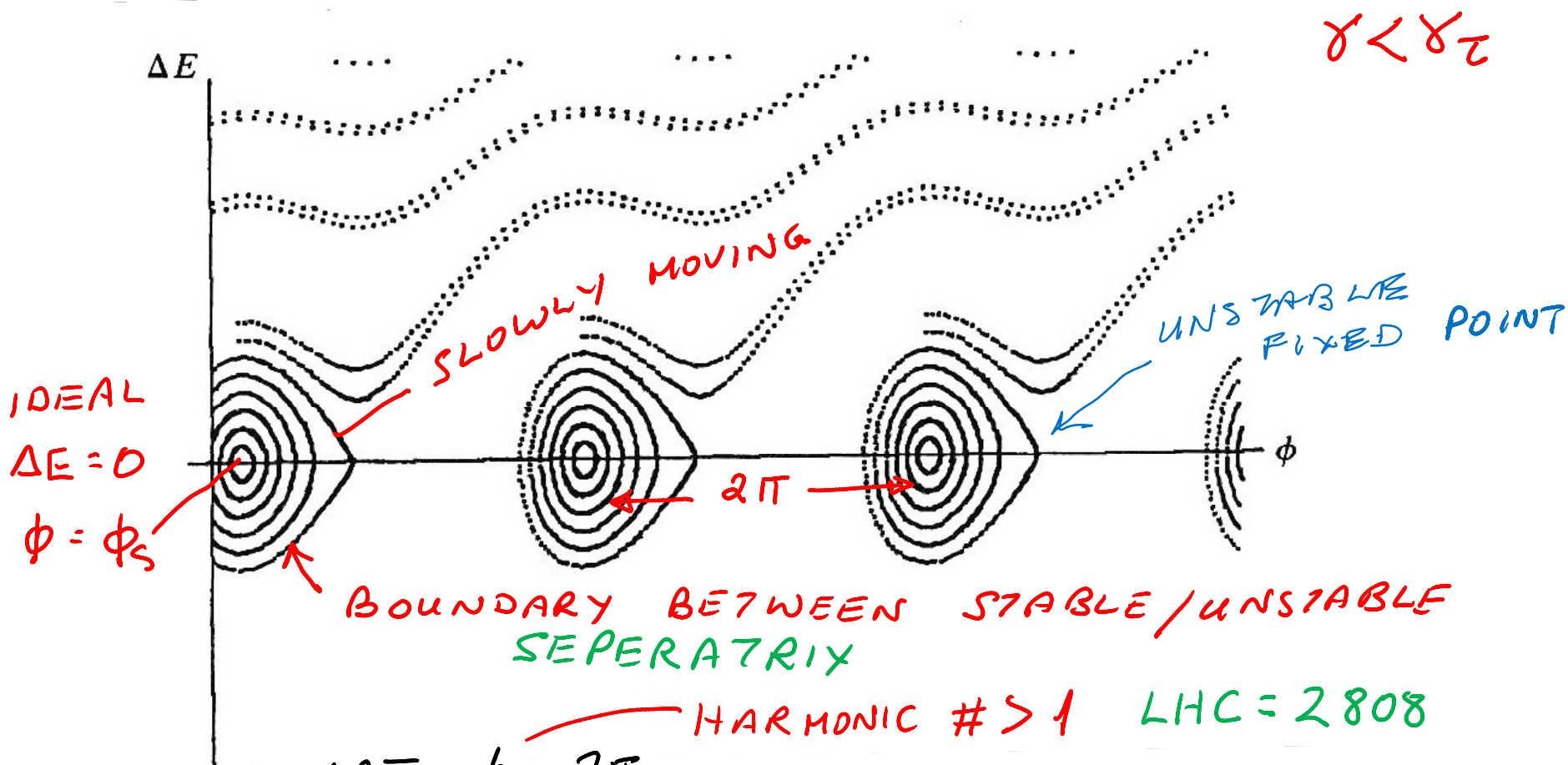


Closed Oscillations in Phase
(non relativistic)

$\delta < \delta_{\tau} \rightarrow \Delta V$ MORE IMPORTANT THAN ΔL

$$(E_s)_{n+1} = (E_s)_n + eV \sin \phi_s \quad \text{Synchronous particle}$$

$$\frac{\Delta \tau}{\tau} = \left(\frac{1}{\gamma_{\tau}^2} - \frac{1}{\gamma^2} \right) \quad \text{Change in transit time around lattice}$$



$\omega\tau = h \cdot 2\pi$

MANY STABLE FIXED POINTS AROUND RING
 SPACED BY 2π

PHASE SPACE WITHIN SEPERATRIX \rightarrow RF BUCKET

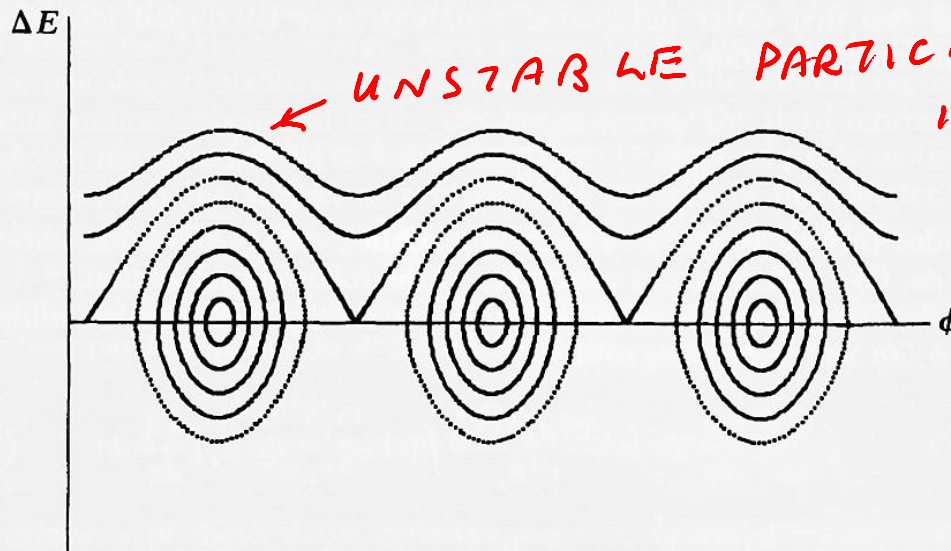


Figure 2.18. Application of the difference equations to a number of initial conditions for $\phi_s = 0$ or π . The regions within the separatrices are called stationary buckets.

• STORAGE RING \rightarrow IDEAL PARTICLE UNACCELERATED

$$\phi_s = 0 \text{ OR } \pi$$

• THIS IS A STORAGE RING WITH NO ENERGY LOSS
RF BUNCHES BEAMS

• FOR ACCELERATION, OR MAKING UP SYNCHROTRON
RADIATION ENERGY LOSS $\phi_s \neq 0$

$$\frac{d\phi}{dn} = \frac{\eta \omega \tau c^2}{v^2 E_s} \cdot \Delta E \quad \leftarrow \quad \frac{d\Delta E}{dn} = eV (\sin \phi - \sin \phi_s)$$

YIELDS

$$\frac{d^2 \phi}{dn^2} = \frac{\eta \omega \tau e V c^2}{v^2 E_s} (\sin \phi - \sin \phi_s)$$

$v \rightarrow$ CONSTANT

dE_s/dn SMALL

CONSTANT

$$\int \frac{d^2 \phi}{dn^2} \cdot \frac{d\phi}{dn} \cdot dn = \frac{\eta \omega \tau e V c^2}{v^2 E_s} \cdot \int (\sin \phi - \sin \phi_s) \frac{d\phi}{dn} \cdot dn$$

$$\frac{1}{2} \left(\frac{d\phi}{dn} \right)^2 = - \frac{\eta \omega \tau e V c^2}{v^2 E_s} (\cos \phi + \phi \sin \phi_s) + C$$

↑
OSCILLATIONS
IN PHASE

↑
ACCELERATION

$$\frac{1}{2} \left(\frac{d\phi}{dn} \right)^2 = - \frac{\eta \omega \tau e V}{v^2 E_s} (\cos \phi + \phi \sin \phi_s) + C$$

$$\left(\frac{\eta \omega \tau c^2}{v^2 E_s} \right)^2 \Delta E^2 = - \frac{2 \eta \omega \tau c^2 e V}{v^2 E_s} (\cos \phi + \phi \sin \phi_s) + C$$

$$\Delta E^2 = - \frac{2 v^2 E_s}{\eta \omega^2 \tau c^2} \cdot e V (\cos \phi + \phi \sin \phi_s) + C$$

↑
KINETIC ENERGY

$$V(\phi) = \frac{2 v^2 E_s e V}{\eta \omega^2 \tau c^2} (\cos \phi + \phi \sin \phi_s)$$

χ

TURNING POINTS IN MOTION FROM

$$\frac{dV}{d\phi} = \chi(-\sin\phi + \sin\phi_s) = 0$$

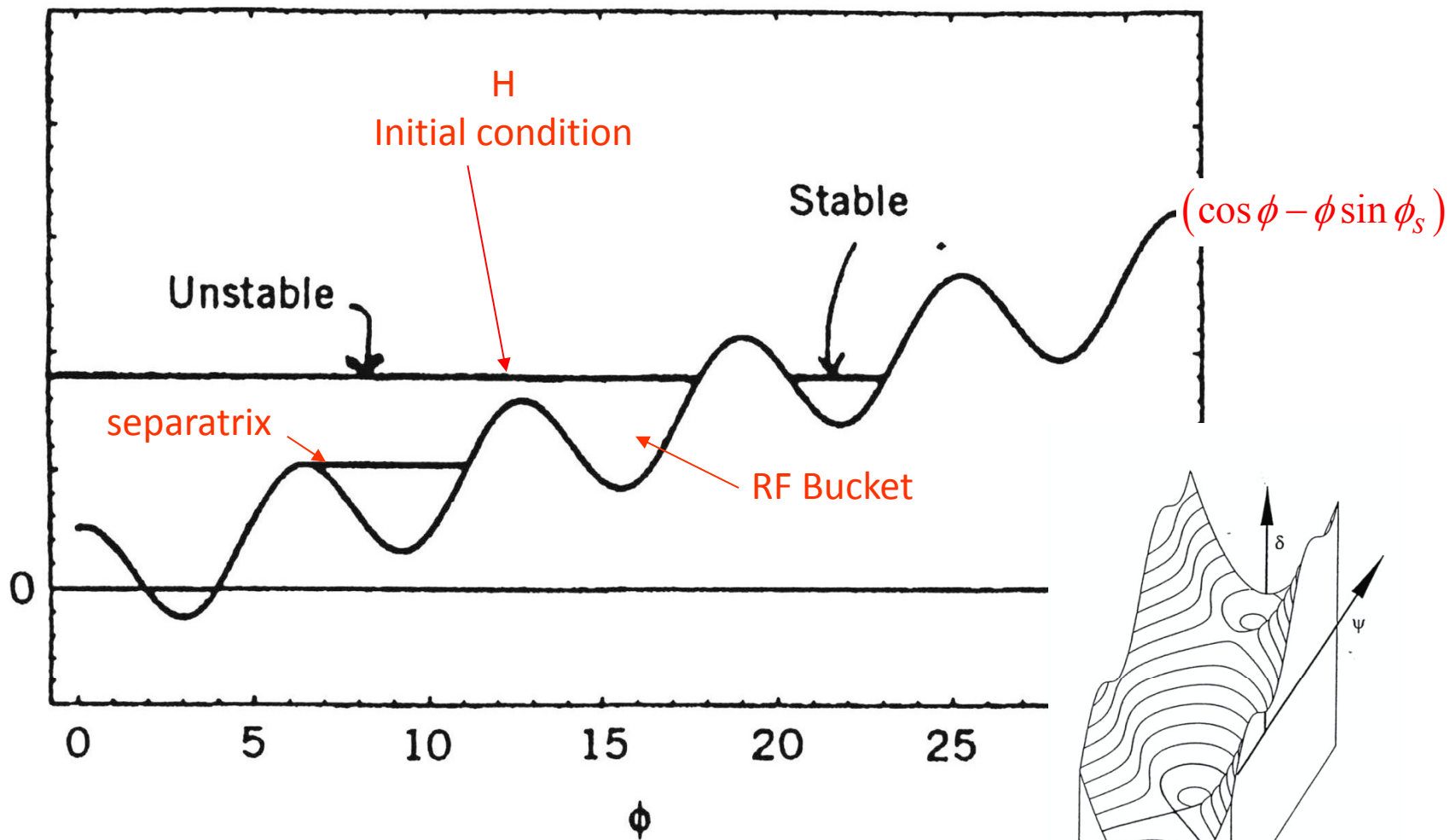
THERE IS A MINIMUM EVERY

$$\phi = \phi_s + 2\pi \quad \leftarrow \text{RF BUCKET}$$

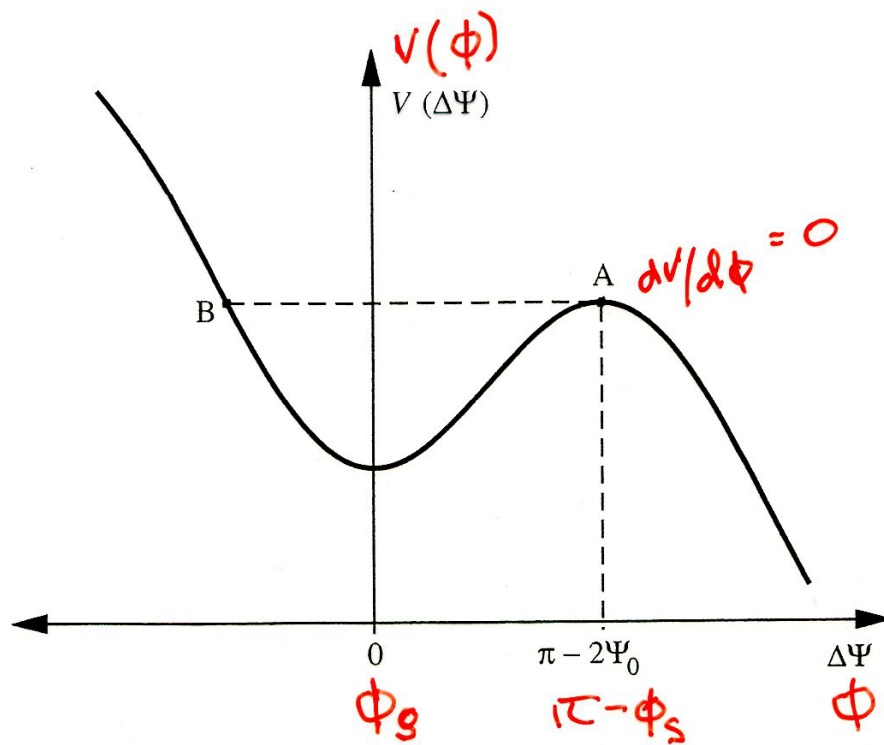
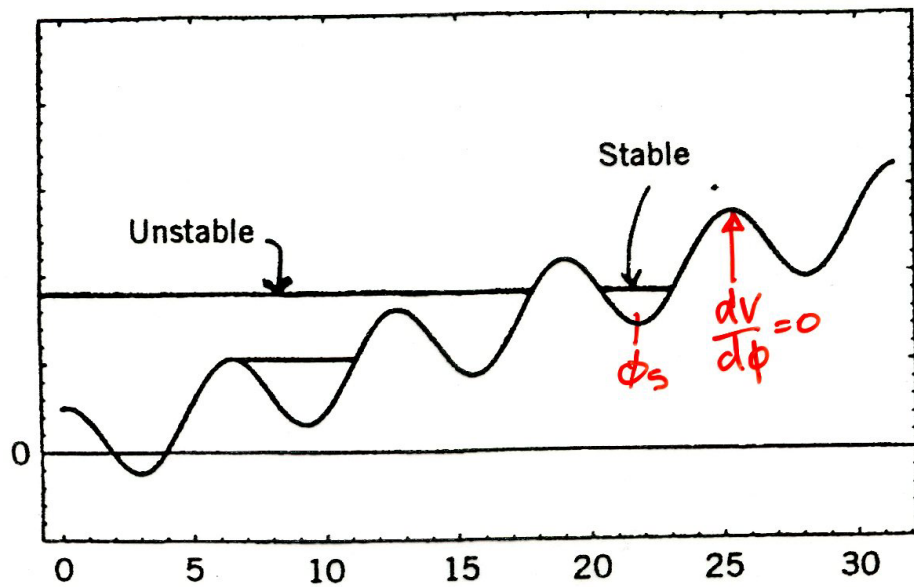
ALSO A TURNING POINT $\phi_{\text{MAX}} = \phi_s - \pi$

STABLE MOTION FOR $\phi < \phi_{\text{MAX}}$

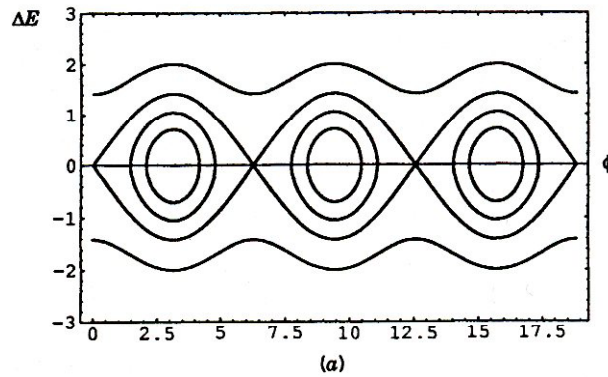
$$V(\phi) \leq V(\phi_{\text{MAX}})$$



$$\text{constant} = H = \frac{1}{2} \left(\frac{d\phi}{dn} \right)^2 + \frac{\eta\omega\tau c^2}{v^2 E_s} (\cos\phi - \phi \sin\phi_s)$$

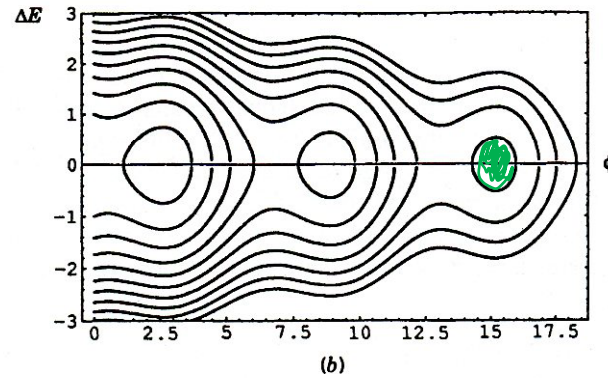


COASTING

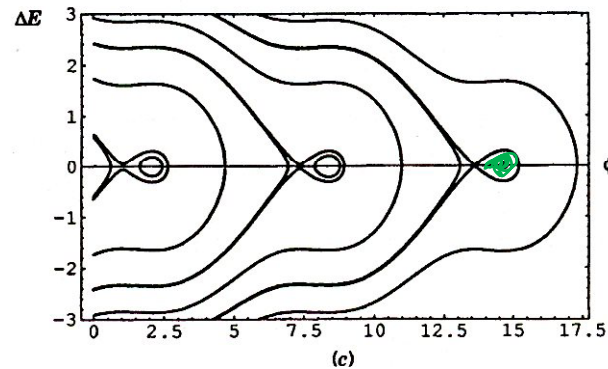


$$\phi_s = \pi$$

ACCELERATING



$$5\pi/6$$



$$2\pi/3$$

Figure 2.20. Contours of particle motion in longitudinal phase space obtained by solving the differential equations of motion. The curves shown are for $\eta > 0$ and for the cases (a) $\phi_s = \pi$, (b) $\phi_s = 5\pi/6$, and (c) $\phi_s = 2\pi/3$.

HAD
$$\frac{d^2 \phi}{dn^2} = \frac{\eta \omega \tau e V c^2}{v^2 E_s} (\sin \phi - \sin \phi_s)$$

ASSUME $\Delta \phi = \phi - \phi_s$ SMALL $\frac{d\Delta \phi}{dn} = \frac{d\phi}{dn}$

$$\begin{aligned} \sin \phi - \sin \phi_s &\approx \sin(\phi_s + \Delta \phi) - \sin \phi_s \\ &= \cos \phi_s \sin \Delta \phi + \sin \phi_s \cos \Delta \phi - \sin \phi_s \\ &\approx \cos \phi_s \Delta \phi \end{aligned}$$

$$\frac{d^2 \Delta \phi}{dn^2} + \left(-\frac{\eta \omega \tau c^2 e V \cos \phi_s}{v^2 E_s} \right) \Delta \phi = 0$$

$$\frac{d^2 \Delta \phi}{dn^2} + (2\pi \nu_s)^2 \Delta \phi = 0$$

SYNCHROTRON OSCILLATIONS
IN ϕ FOR $2\pi \nu_s$ REAL

ν_s = NUMBER OF
SYNCHROTRON
OSCILLATIONS / TURN

$$\frac{d^2 \Delta \phi}{dn^2} + (2\pi \nu_s)^2 \Delta \phi = 0$$

$$\nu_s = \left(\frac{-\eta \omega \tau c^2 eV \cos \phi_s}{4\pi^2 v^2 E_2} \right)^{\frac{1}{2}}$$

STABLE OSCILLATIONS $\nu_s \rightarrow$ REAL

$$\eta = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \right)$$

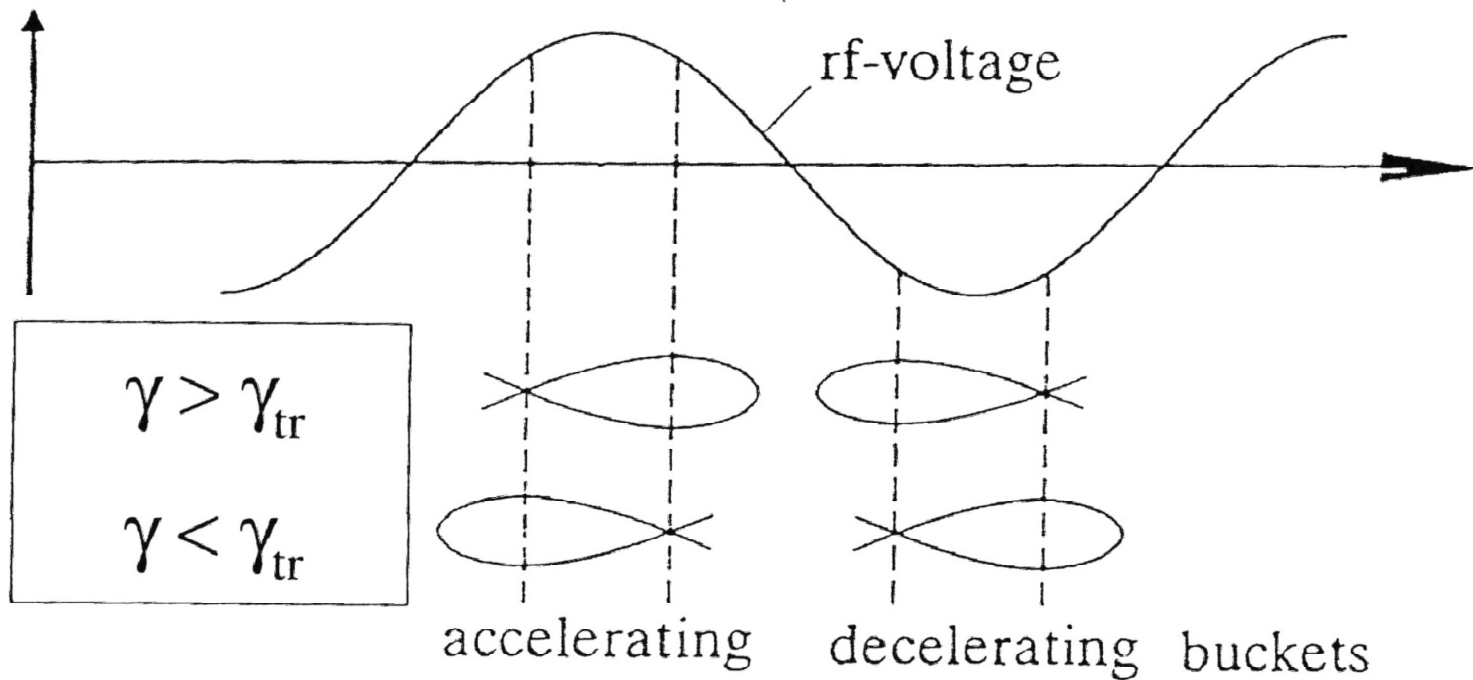
FOR $\eta < 0$ ($\gamma < \gamma_t$) $\cos \phi_s > 0$

FOR $\eta > 0$ ($\gamma > \gamma_t$) $\cos \phi_s < 0$

γ_t IS TRANSITION ENERGY

FOR A CIRCULAR ACCELERATOR AS ENERGY

PASSES THROUGH $\gamma = \gamma_t$ RF FREQUENCY MUST GO THROUGH PHASE SHIFT.



Synchrotron (phase) oscillations

$$\frac{d^2 \Delta\phi}{dn^2} + \left(\frac{-\eta \omega \tau c^2 e V}{v^2 E_s} \cos \phi_s \right) \Delta\phi = 0$$

$$\frac{d^2 \Delta\phi}{dn^2} + (2\pi\nu_s) \Delta\phi = 0$$

$$\eta < 0 \quad ; \quad (\gamma < \gamma_t) \quad ; \quad \cos \phi_s > 0$$

$$\eta > 0 \quad ; \quad (\gamma > \gamma_t) \quad ; \quad \cos \phi_s < 0$$

FOR A LINEAR ACCELERATOR

$$\eta = \frac{1}{\gamma_7^2} - \frac{1}{\gamma^2} \rightarrow \frac{-1}{\gamma^2}$$

THERE IS NO DIFFERENCE IN LENGTH TRAVERSED BY PARTICLES DUE TO $\Delta\phi$

$$V_s \sim \sqrt{\frac{1}{E_s}} \rightarrow 0 \text{ AS } E_s \uparrow$$

FOR $\gamma \gg 1 \rightarrow \Delta\phi = \text{CONSTANT}$.