LINEAR LATTICE DEFECTS

ACCELERATOR IS DESIGNED BY

- CHOOSE DIPOLES TO GIVE CLOSED EQUILIBRIUM ORBIT
- CHOOSE QUADRUPOLES TO GIVE βTRON OSCILLATIONS WHICH FIT IN MACHINE APERTURE.
- CHOOSE SEXTUPOLES TO MAKE MACHINE OPTICS ACHROMATIC.

REAL MAGNETS NOT PERFECT

— DIPOLES DO NOT HAVE DESIGN FIELD

— DIPOLES HAVE QUADRUPOLE COMPONENT

etc

FIRST CONSIDER A DIPOLE WHICH HAS A SLIGHTLY INCORRECT FIELD

? WHAT HAPPENS TO THE EQUILIBRIUM ORBIT ?

SAY ERROR is $\theta = \dfrac{\Delta B \cdot \ell}{B\rho}$ AT $s = 0$

ΔB DIPOLE IMPERFECTION OVER LENGTH ℓ

θ is RESULTING UNEXPECTED BEND

IS THERE STILL A CLOSED EQUILIBRIUM ORBIT AROUND THE MACHINE ?

JUST AFTER IMPERFECTION θ ORBIT IS SPECIFIED BY x_0, x_0'

FOR A CLOSED ORBIT, MUST RETURN TO x_0 x_0' AFTER ONE TURN AROUND THE MACHINE

$$M\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

MATRIX FOR SINGLE TURN

ANGLE INTRODUCED BY LATTICE IMPERFECTION

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \left(\underline{I} - M\right)^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

USE $M = e^{J\Delta\phi}$

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

PHASE CHANGE $= 2\pi\nu$

$$\left(\underline{I} - M\right)^{-1} = \left(\underline{I} - e^{J2\pi\nu}\right)^{-1}$$

$$\left(I - e^{J2\pi\nu}\right)^{-1} = \left[e^{J\pi\nu}\left(e^{-J\pi\nu} - e^{+J\pi\nu}\right)\right]^{-1}$$

$$= -\left(2J\sin\pi\nu\right)^{-1}\left(e^{J\pi\nu}\right)^{-1}$$

$$J^2 = -I \quad ; \quad J = \frac{-I}{J} \quad ; \quad J = \sqrt{I}$$

$$= \frac{J}{2\sin\pi\nu} \cdot e^{-J\pi\nu} = \frac{J}{2\sin\pi\nu}\left\{I\cos(-\nu\pi) + J\sin(-\pi\nu)\right\}$$

$$= \frac{1}{2\sin\pi\nu}\left\{J\,I\cos(\pi\nu) - J^2\sin(\pi\nu)\right\}$$

$$= \frac{1}{2\sin\pi\nu}\left\{J\cos(\pi\nu) + I\sin(\pi\nu)\right\}$$

CLOSED ORBIT AT $\delta = 0$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$= \frac{1}{2\sin\pi\nu} \left\{ \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \cos\pi\nu + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sin\pi\nu \right\} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0 \cos\pi\nu \end{pmatrix}$$

$$M_S = \begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{1/2}\left(\cos\Delta\phi_c + \alpha_1 \sin\Delta\phi_c\right) & \left(\beta_1\beta_2\right)^{\frac{1}{2}} \sin\Delta\phi_c \\ -\frac{1+\alpha_1\alpha_2}{\left(\beta_1\beta_2\right)^{\frac{1}{2}}}\sin\Delta\phi_c + \frac{\alpha_1-\alpha_2}{\left(\beta_1\beta_2\right)^{\frac{1}{2}}}\cos\Delta\phi_c & \left(\frac{\beta_1}{\beta_2}\right)^{1/2}\left(\cos\Delta\phi_c - \alpha_2 \sin\Delta\phi_c\right) \end{pmatrix}$$

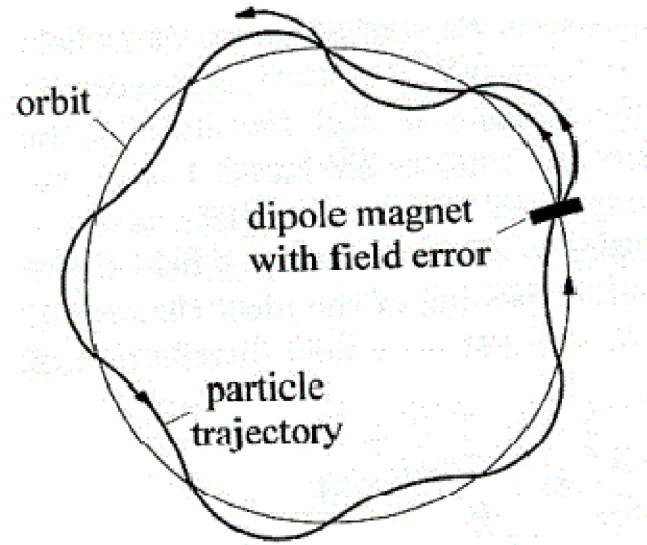
IS THE MATRIX FOR $1 \to 2$

POSITION ON PERTURBED ORBIT $\begin{pmatrix} x_s \\ x_s' \end{pmatrix} = M_s \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

$$\begin{pmatrix} x_s \\ x_s' \end{pmatrix} = M_s \cdot \frac{\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

$$x(s) = \frac{\theta\, \beta(s)^{1/2} \cdot \beta_0^{1/2}}{2\pi \sin\pi\nu} \cdot \cos\underbrace{\left(\phi(s) - \pi\nu\right)}_{\Delta\phi}$$

$$x(s) = \frac{\theta\, \beta^{1/2}(s) \cdot \beta_0^{1/2}}{2\pi \sin \pi\nu} \cdot \cos(\phi(s) - \pi\nu)$$

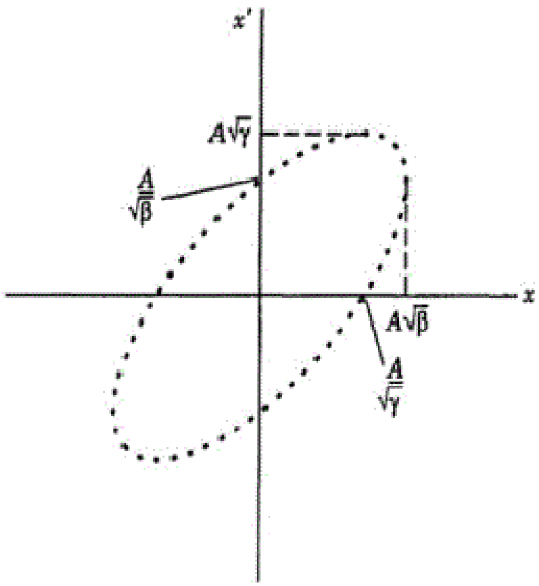


WITH DIPOLE ERROR
STILL HAVE CLOSED ORBIT
WITH A KINK AT ERROR.

NOTE THAT FOR ν = INTEGER
– INTEGRAL NUMBER OF
BETATRON OSCILLATIONS AROUND
RING $x(s) \to \infty$

RESONANCE → IMPERFECTION GIVES ORBIT
SAME KICK AT SAME $x(s)$ EACH TIME
AROUND. $x(s)$ GROWS UNTIL HIT BEAM
TUBE, AND LOSE BEAM.

RESONANCES & CHAOTIC MOTION



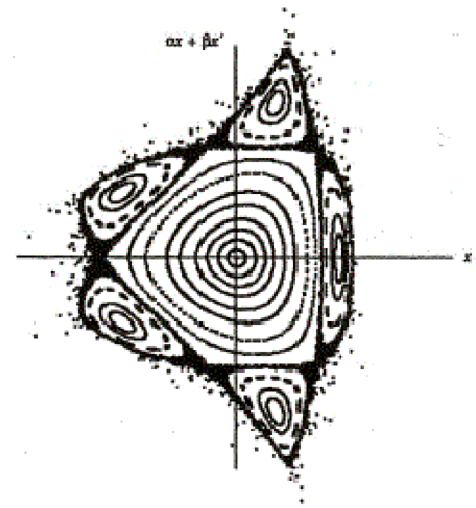


Figure 4.1. Phase space plot of particles circulating in a synchrotron lattice that is perfectly linear except for a single thin sextupole field introduced at the point of observation. For this plot, the tune of the accelerator is 0.42 plus an integer.

FIELD IMPERFECTIONS INTRODUCE NON LINEARITIES

THEY GROW WITH AMPLITUDE → CHAOTIC

MAKE RUNNING A REAL ACCELERATOR A COMPLEX TASK.

DRIVEN OSCILLATOR — RESONANCES

SOLUTION TO HILL'S EQUATION

$$x(s) = A\beta^{\frac{1}{2}}\cos[\psi + \delta]$$

DEFINE REDUCED PHASE $\phi = \psi/\nu$

ϕ INCREASES BY 2π EACH TIME AROUND THE RING

$$\phi(s) = \frac{\psi(s)}{\nu} = \frac{1}{\nu}\oint \frac{ds}{\beta(s)}$$

REPLACE TRANSVERS COORDINATE $x(s)$ BY $\eta(s) = x(s)/\beta^{\frac{1}{2}}$

$$\left.\begin{aligned} &\eta(s) = A\cos(\nu\phi + \delta) \\ &\frac{d^2\eta}{d\phi^2} + \nu^2\eta = 0 \end{aligned}\right\}$$

SIMPLE HARMONIC MOTION IN NEW COORDINATES

FLOQUET'S TRANSFORMATION

$$\frac{d\eta}{d\phi} = \frac{d\eta}{ds} \cdot \frac{ds}{d\phi} = \frac{d}{ds}\left(\frac{x(s)}{\sqrt{\beta(s)}}\right)\nu\beta(s)$$

$$= \left(\frac{\alpha(s)}{\sqrt{\beta(s)}} \cdot x(s) + \sqrt{\beta(s)} \cdot x'(s)\right) \cdot \nu$$

AND

$$\frac{d^2\eta}{d\phi^2} = \frac{d}{d\phi}\left(\frac{d\eta}{d\phi}\right) = \frac{d}{ds}\left(\frac{d\eta}{d\phi}\right)\nu\beta(s)$$

$$= \left\{\beta^{3/2}(s)\,x''(s) + \frac{\alpha^2(s)}{\sqrt{\beta}} \cdot x(s) + \alpha'(s)\sqrt{\beta}\,x(s)\right\}\nu^2$$

FROM DEFINITIONS OF α, β, γ

$$\alpha'(s) = \beta(s)K(s) - \gamma(s)$$

$$\beta'(s) = -2\alpha(s)$$

$$\gamma'(s) = 2\alpha(s)K(s)$$

$$x''(s) + K(s)x(s) = 0$$

$$\frac{d^2\eta}{d\phi^2} = \left\{ \beta^{3/2} x'' + \alpha^2 \frac{x}{\sqrt{\beta}} + \left[K\beta - \frac{1+\alpha^2}{\beta} \right] \sqrt{\beta} \cdot x \right\} \nu^2$$

$$= \left\{ \beta^{3/2} \left[x'' + K \cdot x \right] - \frac{x}{\sqrt{\beta}} \right\} \nu^2$$

MULT $x''(s) + K(s)\,x(s) = 0$ BY $\beta^{3/2} \nu^2$

$$\left[x''(s) + K(s)\,x(s) \right] \beta^{3/2} \nu^2 = 0$$

$$= \underbrace{\left\{ \beta^{3/2} \left[x'' + K \cdot x \right] - \frac{x}{\sqrt{\beta}} \right\} \nu^2}_{d^2\eta / d\phi^2} + \frac{x}{\sqrt{\beta}} \nu^2$$

$$\frac{d^2\eta}{d\phi^2} + \nu^2 \eta = 0$$

FLOQUET'S TRANSFORMATION

IN AN IDEAL LINEAR MACHINE
MAGNETIC FIELD

$$\frac{e}{p} B_y(x,s) = \frac{e}{p} B_{y_0}(s) + \frac{e}{p} \frac{dB_y}{dx} \cdot x(s)$$

$$= \frac{1}{\rho} + k(s) \cdot x(s)$$

LOOK BACK A FEW LECTURES

THIS IS EXACTLY THE FIELD THAT GAVE US

$$\ddot{x}(s) + \left[\frac{1}{\rho^2} + \frac{1}{B\rho} k(s)\right] x(s) = 0$$

$\frac{dB_y}{dx}$

AND THE HOMOGENEOUS HILL'S EQUATION

$$\ddot{x}(s) + K(s) \cdot x(s) = 0$$

NOW WE ADD A DIPOLE IMPERFECTION

$$\frac{e}{p}\tilde{B}_y(x,s) = \frac{1}{\rho} + k(s)x(s) + \frac{e}{p}\Delta B(x,s)$$

$$= \frac{1}{\rho} + k(s)x(s) + \frac{\Delta B}{\rho B_{y_0}}$$

THIS FIELD GIVES US INHOMOGENEOUS HILL'S

$$x''(s) + K(s)x(s) = \frac{\Delta B}{\rho B_y} \xrightarrow{\text{FLOQUET}} \frac{d^2\eta}{d\phi^2} + \nu^2\eta = \beta^{3/2}\frac{\nu^2\Delta B}{B\rho}$$

PERIODIC SOLUTION

$$\eta(\phi) = \frac{\nu}{2\sin\pi\nu}\int_{\phi}^{\phi+2\pi}\beta^{3/2}\frac{\Delta B}{\rho B}\cos\left[\nu(\pi+\phi+\theta)\right]d\theta$$

RESONANCE FOR ν = INTEGER

EXPAND FIELD ERROR IN MULTIPOLES

$$\Delta B(x) = \Delta B_0 + \frac{d\Delta B}{dx}\cdot x + \frac{1}{2}\frac{d^2\Delta B}{dx^2} + \frac{1}{3}\frac{d^3\Delta B}{dx^2} + \cdots$$

RECALL $\frac{d}{d\eta} = \frac{d}{dx}\frac{dx}{d\eta}$ AND $\frac{dx}{d\eta} = \sqrt{\beta}$

$$\Delta B(\eta) = \Delta B_0 + \beta^{\frac{1}{2}}\frac{d\Delta B}{d\eta} + \frac{1}{2}\beta^{\frac{2}{2}}\frac{d^2\Delta B}{d\eta^2}\cdot\eta^2 + \cdots$$

PUT THIS INTO $\frac{d^2\eta}{d\phi^2} + \nu^2\eta = \beta^{3/2}\nu^2\frac{\Delta B}{\rho B}$

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = \frac{\nu^2}{\rho B}\left(\beta^{3/2}\Delta B + \beta^{4/2}\frac{d\Delta B}{d\eta}\eta + \frac{1}{2}\beta^{5/2}\frac{d^2\Delta B}{d\eta^2}\eta^2 + \cdots\right)$$

OSCILLATOR DRIVEN BY MULTIPOLES

$$\frac{d^2\eta}{d\phi^2}+\nu^2\eta=\frac{\nu^2}{\rho B}\left(\underbrace{\beta^{3/2}\Delta B}+\beta^{4/2}\frac{d\Delta B}{d\eta}\eta+\frac{1}{2}\beta^{5/2}\frac{d^2\Delta B}{d\eta^2}\eta^2+\cdots\right)$$

THESE TERMS CAN BE WRITTEN AS FOURIER SERIES

$$\left(\beta^{(n+3)/2}\frac{d^n\Delta B}{d\eta^n}\right)=\sum_k c_k e^{\pm ik\phi}$$

AND SOLUTION OF EQUATION OF MOTION

$$\eta(\phi)=\eta_0 e^{\pm i\nu\phi}$$

$k=\nu$ INTEGER RESONANCE

$k-\nu=\nu \rightarrow k=2\nu$ $\frac{1}{2}$ INTEGER RESONANCE

$k-2\nu=\nu \rightarrow k=3\nu$ $\frac{1}{3}$ INTEGER RESONANCE

DRIVING FIELD	CONDITION
DIPOLE	$\nu = p$
QUADRUPOLE	$2\nu = p$
SEXTUPOLE	$3\nu = p$

HAVE RESONANCES IN BOTH PLANES IN A REAL ACCELERATOR → COUPLED

RESONANCE CONDITION IS:

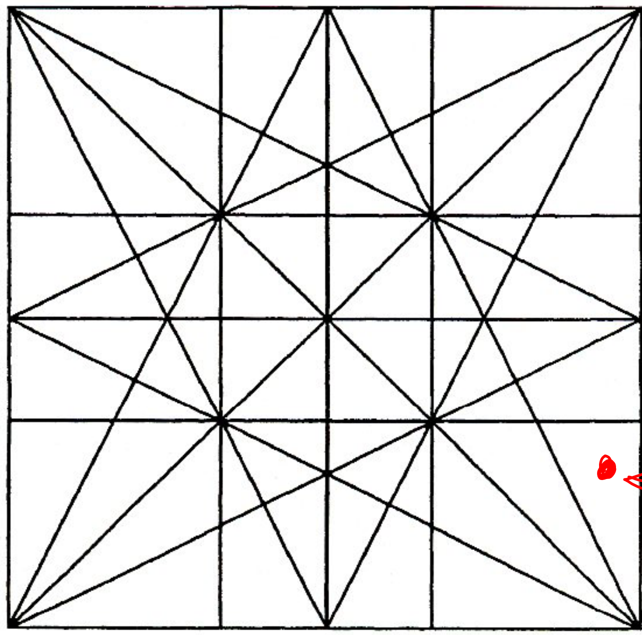
$$M\nu_x + N\nu_y = P$$

EG SEXTUPOLE

$$\left.\begin{aligned} 3\nu_x &= p \\ 2\nu_x + \nu_y &= p \\ \nu_x + 2\nu_p &= p \\ 3\nu_y &= p \end{aligned}\right\}$$

GIVE UNSTABLE RESONANCES

MACHINE DOES NOT WORK!



NO STABLE BEAM IF LAND ON RESONANCE LINE

POSSIBLE TUNE GIVING STABLE BEAM

Figure 5.7. Resonance lines in a unit square of the tune plane for third integer and below.

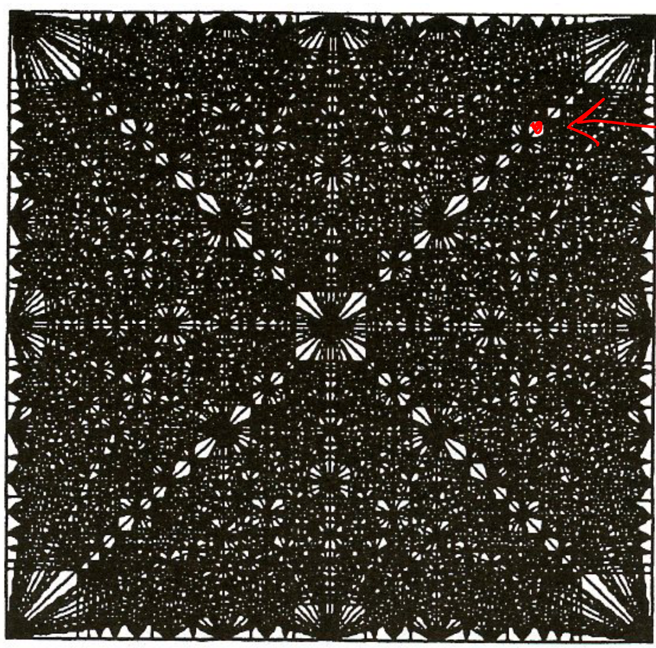


Figure 5.8. Resonance lines in the tune plane for ninth integer and below.