LUMINOSITY

INTERACTION RATE IN A COLLIDER

$$R = \frac{dN_p}{dt} = \sigma_p \mathcal{L}$$ ← LUMINOSITY

$$\mathcal{L}\left[\frac{1}{cm^2 \cdot s}\right] = \mathcal{L}\left[\frac{10^{33}}{nb \cdot s}\right] = \frac{1}{\sigma_p}\frac{dN_p}{dt}$$

IN MODERN EXPERIMENTS $\sigma_p \ll 1\,nb$

NEED VERY HIGH LUMINOSITY

$$N_p = \sigma_p \int \mathcal{L}\,dt = \sigma_p \mathcal{L}_{INT}$$

INTEGRATED LUMINOSITY

THIS DIAGRAM SAY e^+e^-
BUT IT IS COMPLETELY GENERAL

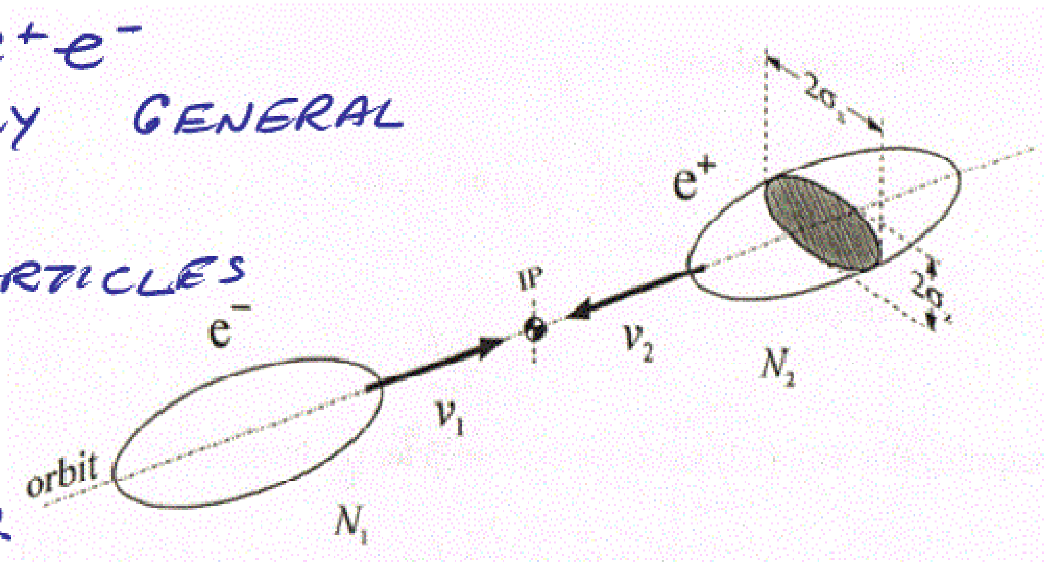


Fig. 7.1 Particle bunches colliding at the interaction point (IP).

ASSUME THAT THE DISTRIBUTION OF PARTICLES IS GAUSSIAN IN ALL DIRECTIONS.

SURFACE DESITY OF N_2

THE TWO BUNCHES PASS COMPLETELY THROUGH EACH OTHER. SO CAN PROJECT ONTO xz PLANE – 2d PROBLEM

$$n_2 = \frac{\partial^2 N_2}{\partial x \, \partial y} = \frac{N_2}{2\pi \sigma_x^* \sigma_z^*} \exp\left(\frac{-x^2}{2\sigma_x^{*2}} - \frac{z^2}{2\sigma_z^{*2}} \right)$$

N_2 TOTAL NUMBER OF PARTICLES IN BUNCH

σ_x^* σ_z^* BEAM WITH AT INTERACTION POINT IP

CONVENTIONALLY IP QUANTITIES $\rightarrow$ *

IF THE BEAMS HAVE THE SAME CROSS SECTION AND OVERLAP COMPLETELY

PROBABILITY OF A PARTICLE IN BUNCH 1 IN A SURFACE ELEMENT $dA = dx\,dz$ INTERACTING WITH A PARTICLE IN BUNCH 2

$$dW = \sigma_p \frac{n_2\,dx\,dz}{dA} = \sigma_p n_2$$

RATE OF BUNCH 1 PARTICLES CROSSING dA

$$\frac{dN_1}{dt} = \frac{b f_{REV} N_1}{2\pi\sigma_x^* \sigma_z^*} \exp\left(-\frac{x^2}{2\sigma_x^{*2}} - \frac{z^2}{2\sigma_z^{*2}}\right) dx\,dz$$

b EQUALLY SPACED BUNCHES, FREQUENCY f_{REV}

TOTAL INTERACTION RATE

$$\frac{dN_p}{dt} = \sigma_p \frac{dN_1}{dt} n_2 = \sigma_p \frac{b f_{REV} N_1 N_2}{(2\pi)^2 \sigma_x^{*2} \sigma_z^{*2}} \exp\left(-\frac{x^2}{\sigma_x^{*2}} - \frac{z^2}{\sigma_z^{*2}}\right) dx\,dz$$

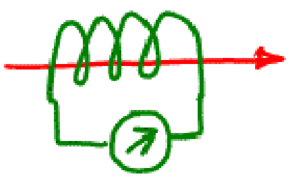
$$\frac{dN_p}{dt} = \sigma_p \frac{dN_1}{dt} n_2 = \sigma_p \frac{b\, f_{REV}\, N_1 N_2}{(2\pi)^2 \sigma_x^{*2} \sigma_z^{*2}} \exp\left(-\frac{x^2}{\sigma_x^{*2}} - \frac{z^2}{\sigma_z^{*2}}\right) dx\, dz$$

CAN USE $\displaystyle\int_{-\infty}^{+\infty} \exp\left(-y^2/\sigma^2\right) dy = \sqrt{\pi}\,\sigma$

TOTAL INTERACTION RATE

$$\frac{dN_p}{dt} = \sigma_p \frac{b\, f_{REV}\, N_1 N_2}{4\pi \sigma_x^* \sigma_y^*} \rightarrow \mathcal{L} = \frac{b}{4\pi} \frac{N_1 N_2}{\sigma_x^* \sigma_y^*} \cdot f_{REV}$$

EASY TO MEASURE AVERAGE BEAM CURRENT



$$I = N e\, f_{REV}\, b$$

LARGE

$$\mathcal{L} = \frac{1}{4\pi e^2 f_{REV} \cdot b} \frac{I_1 I_2}{\sigma_x^* \sigma_y^*}$$

SMALL

BEAM-BEAM INTERACTIONS & LUMINOSITY

HAVE SPACE CHARGE
INTERACTION BETWEEN BEAMS

GO INTO CENTRE OF MASS
OF BUNCH ② $- K'$

BUNCH ② ONLY HAS
ELECTRIC FIELD $\bar{E}'$

IN LAB FRAME $\bar{B} + \bar{E}$

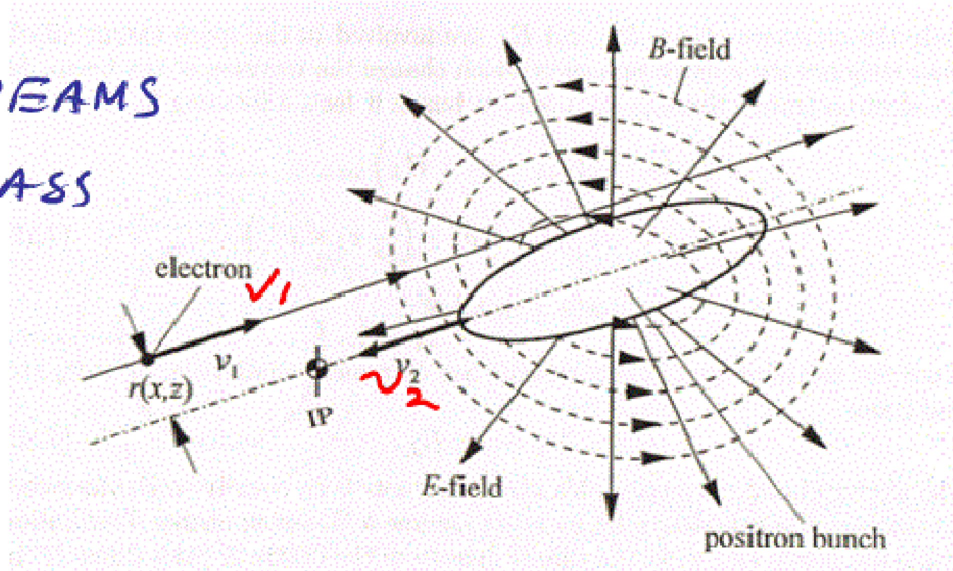


Fig. 7.2 Deflection of an electron due to the space charge of an oncoming bunch.

$$\bar{E}_\perp = \gamma \bar{E}'_\perp \qquad \bar{E}_{\parallel} = \bar{E}'_{\parallel}$$

$$\bar{B}_\perp = \frac{\gamma}{c^2} \bar{v}_2 \times \bar{E}'_\perp \qquad B_{\parallel} = 0$$

ONLY $\bar{B}_\perp$ $\bar{E}_\perp$
GIVE FORCE ON BUNCH ①
PASSING THRU BUNCH ②

FORCE ON ① $= \bar{F}_\perp = -e\left(\bar{E}_\perp + \bar{v}_1 \times B_\perp\right)$ IN LAB

$$= -e\left[\gamma E'_\perp + \bar{v}_1 \times \bar{B}_\perp\right] = -e\left[\gamma E'_\perp + \bar{v}_1\left(\frac{\gamma}{c^2}\bar{v}_2 \times E'_\perp\right)\right]$$

$$= -e\left(1 + \frac{v_1 v_2}{c^2}\right)E_\perp \qquad \frac{v}{c} \to 1 \qquad \boxed{\bar{F}_\perp = -2e E_\perp}$$

ASSUME BEAMS GAUSSIAN IN ALL 3 DIRECTIONS IN CM FRAME OF ①, CHARGE DENSITY IN ② SEEN BY ① $\rho(x', y', s') = \rho(x, y, s')$

BOOST IS ALONG s

$$\rho'(x, y, s') = \frac{eN_2}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_s'} \exp\left\{ \frac{-x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2} - \frac{(s' - s_0')^2}{2\sigma_s'^2} \right\}$$

$\sigma_{x,z}' = \sigma_{x,z}$ $\sigma_s' = \gamma\sigma_s$, s_0' IS ARBITRARY REFERENCE POINT

$\sigma_s' = \gamma\sigma_s \ggg \sigma_x, \sigma_y \rightarrow$ ONLY LONG CHARGE DISTRIB

$$\rho'(x, z, s') = A(s') \exp\left(\frac{-x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2} \right)$$

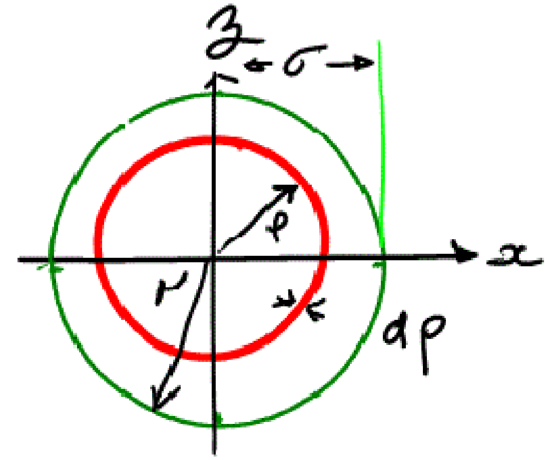
WITH

$$A(s') = \frac{eN_2}{(2\pi)^{3/2}\sigma_x \sigma_z \sigma_s'} \exp\left(- \frac{(s' - s_0')}{2\sigma_s'^2} \right)$$

USE THIS CHARGE DISTRIBUTION TO CALC FORCE ON ① SIMPLIFY $\sigma = \sigma_x = \sigma_y$ $r^2 = x^2 + z^2$

SIMPLIFIED CHARGE DENSITY, SYMMETRICAL BEAMS

$$\rho'(r, s') = A(s') \exp\left(\frac{-r^3}{2\sigma^2}\right)$$



CHARGE ON SHELL $\rho \, d\rho$

$$dq = \rho'(r, s') \, 2\pi \rho \, d\rho \, ds'$$

CYLINDER LENGTH $\Delta s'$

$$\Delta q(s') = 2\pi A(s') \, \Delta s' \int_0^r \exp\left(\frac{-\rho^2}{2\sigma^2}\right) \rho \, d\rho$$

MAPLE → $$\Delta q(s') = 2\pi A(s') \sigma^2 \left[1 - \exp\left(\frac{-r^2}{2\sigma^2}\right)\right] \Delta s'$$

GAUSS'S THEOREM ON CYLINDRICAL SURFACE

$$E'_\perp(r) \, 2\pi r \, \Delta S' = \Delta q(s') / \varepsilon_0$$

$$E'_\perp(r) = \frac{\Delta q(s')}{2\pi \varepsilon_0 r \, \Delta s'}$$ ← PUT IN $\Delta q(s')$ AND $A(s')$

$$E'_\perp(r) = \frac{e N_2}{(2\pi)^{3/2}\varepsilon_0 r \sigma'_s} \exp\left(-\frac{(s'-s'_0)^2}{2\sigma_s'^2}\right)\left[1-\exp\left(\frac{-r^2}{2\sigma^2}\right)\right]$$

$$s' = \gamma s \qquad \sigma_s' = \gamma \sigma_s$$

$$E_\perp(r,s) = \gamma E'_\perp(r,s) = \frac{e N_2}{(2\pi)^{3/2}\varepsilon_0 \sigma_s} \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right)\frac{1}{r}\left[1-\exp\left(\frac{-r^2}{2\sigma^2}\right)\right]$$

ONLY CONSIDER 1 AT VERY SMALL DISTANCE FROM BEAM AXIS PASSING THROUGH 2

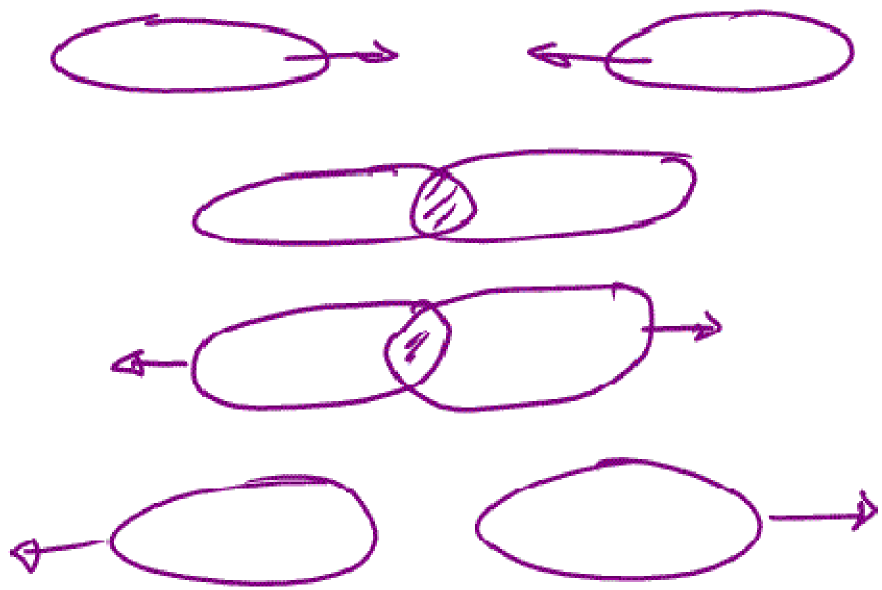
$$\exp\left(\frac{-r^2}{2\sigma^2}\right) = 1 - \frac{r^2}{2\sigma^2} + \frac{1}{2!}\left(\frac{r^2}{2\sigma^2}\right)^2 \cdots \approx 1 - \frac{r^2}{2\sigma^2}$$

$$E_\perp(r,s) = \frac{e N_2}{(2\pi)^{3/2}\varepsilon_0 \sigma_s} \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right)\frac{r}{2\sigma^2}$$

PARTICLE 1 IS ACCELERATED BY $E_\perp$ CHANGES TRANSVERSE MOMENTUM

$$dp_\perp = F_\perp\, dt = F_\perp \frac{ds}{2c}$$

WHERE DOES THE FACTOR OF $\frac{1}{2}$ REALLY COME FROM?



FORCE ONLY FELT WHILE BUNCHES COLLIDING AS SOON A "TEST PARTICLE" HAS TRAVELED $\frac{1}{2}$ BUNCH LENGTH $\rightarrow$ THE ONCOMING BUNCH HAS GONE PAST.

$$dp_\perp = F_\perp \frac{ds}{2c} = -2eE_\perp \frac{ds}{2c}$$

$$dp_\perp = -\frac{e^2 N_2 r}{4\pi\varepsilon_0 c\sigma^2} \frac{1}{\sqrt{2\pi}\,\sigma_s} \cdot \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right) ds$$

GAUSSIAN IN S DIRECTION DIES AWAY FOR $|s-s_0|$ LARGE $\therefore \int -\infty \to \infty$

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right) ds = \sqrt{2\pi}\,\sigma_s$$

$$\Delta p_\perp(r) = -\frac{e^2 N_2}{2\pi\varepsilon_0 c} \cdot \frac{r}{2\sigma^2} \rightarrow \Delta x' = \frac{\Delta p_x}{p} = -\frac{e^2 N_2}{2\pi p c \varepsilon_0} \frac{x}{2\sigma^2}$$

$$\Delta z = \frac{\Delta p_z}{p} = -\frac{e^2 N_2}{2\pi p c \varepsilon_0} \frac{z}{2\sigma^2}$$

$$\Delta x' = -\frac{e^2 N_2}{2\pi p c \varepsilon_0} \frac{x}{2\sigma^2}$$

FOR A MAGNETIC QUADRUPOLE LENS

$$\Delta x' = k \cdot l \cdot x$$

SO BUNCH (2) ACTS ON (1) AS QUADRUPOLE AND VICE VERSA – FOCUS/DEFOCUS DEPENDS ON WHETHER BEAMS ARE SAME/OPPOSITE SIGN.

QUADRUPOLE STRENGTH $k_r \cdot l = -\frac{e^2}{2\pi} \frac{N_2}{p c \varepsilon_0} \frac{1}{2\sigma^2}$

BUT TUNE SHIFT

$$\Delta \nu = \frac{1}{4\pi} \int_{s_0}^{s_0 + l} \Delta k \underset{\uparrow \atop \beta \text{ fn}}{\beta(s)}\, ds$$

$$\left.\begin{aligned} \Delta \nu_{x,z} &= \frac{\beta^*_{x,z}}{4\pi} k_r \cdot l \\ \Delta \nu_{x,z} &= \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \cdot \frac{\beta^*_{x,z}}{2\sigma^2} \end{aligned}\right.$$

$$\Delta \nu_{x,z} = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \frac{\beta^*_{x,y}}{2\sigma^2}$$ ← β FN AT INTERACTION POINT

BEAM – BEAM TUNE SHIFT

AS NUMBER OF PARTICLES IN A BUNCH INCREASES TUNE SHIFT INCREASES UNTIL BEAM MOVES ONTO RESONANCE AND IS LOST

FOR ELLIPTICAL CROSS SECTION

$$2\sigma^2 \rightarrow \quad \sigma_x^*(\sigma_x^* + \sigma_z^*)$$
$$\sigma_z^*(\sigma_x^* + \sigma_z^*)$$

$$\Delta \nu_x = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \frac{\beta_x^*}{\sigma_x^*(\sigma_x^* + \sigma_z^*)}$$

$$\Delta \nu_z = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \frac{\beta_z^*}{\sigma_z^*(\sigma_x^* + \sigma_z^*)}$$

$$N_2 = \frac{I_2}{b f_{rev} e} \qquad \sigma^*_{x,z} = \sqrt{\varepsilon_{x,z} \beta^*_{x,z}}$$

REMEMBER HILL'S EQUATION !

$$E = pc \;, \quad \varepsilon_0 = 1/\mu_0 c^2$$

$$\Delta \nu_x = \frac{\mu_0 e c^2 I_2}{8\pi^2 b f_{rev} E} \cdot \frac{\sqrt{\beta^*_x}}{\sqrt{\varepsilon_x}\left(\sqrt{\varepsilon_x \beta^*_x} + \sqrt{\varepsilon_z \beta^*_z}\right)}$$

$$\Delta \nu_z = \frac{\mu_0 e c^2 I_2}{8\pi^2 b f_{rev} E} \cdot \frac{\sqrt{\beta^*_z}}{\sqrt{\varepsilon_z}\left(\sqrt{\varepsilon_x \beta^*_x} + \sqrt{\varepsilon_z \beta^*_z}\right)}$$

TYPICALLY $\Delta \nu \approx 0.2$

SO TUNE SHIFT RESTRICTS MAXIMUM CURRENT (AND HENCE LUMINOSITY)

$$I_{MAX,x} = \frac{8\pi^2 b f_{REV} E \sqrt{\varepsilon_x}\left(\sqrt{\varepsilon_x \beta_x^*} + \sqrt{\varepsilon_z \beta_z^*}\right)}{\mu_0 e c^2 \sqrt{\beta_x^*}} \cdot \Delta\nu_x^{MAX}$$

$$I_{MAX,z} = \frac{8\pi^2 b f_{REV} E \sqrt{\varepsilon_z}\left(\sqrt{\varepsilon_x \beta_x^*} + \sqrt{\varepsilon_z \beta_z^*}\right)}{\mu_0 e c^2 \sqrt{\beta_z^*}} \cdot \Delta\nu_z^{MAX}$$

HORIZONTAL – VERTICAL COUPLING OF BTRON OSC

$$k = \frac{\varepsilon_z}{\varepsilon_x} \rightarrow \quad \varepsilon_x = \frac{\varepsilon_{x_0}}{1+k}, \quad \varepsilon_z = \frac{k\,\varepsilon_{x_0}}{1+k}$$

IDEAL MACHINE $k = 0$ BUT COUPLING

DRIVES VERTICAL OSCILLATION

$$I_{MAX,x} = \frac{8\pi^2 b f_{REV} E \varepsilon_{x_0} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)}{\mu_0 e c^2 (1+k) \sqrt{\beta_x^*}} \cdot \Delta\nu_x^{MAX}$$

$$I_{MAX,z} = \frac{8\pi^2 b f_{REV} E \varepsilon_{x_0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)}{\mu_0 e c^2 (1+k) \sqrt{\beta_z^*}} \Delta\nu_z^{MAX}$$

THESE ARE THE MAXIMUM CURRENTS ALLOWED BY x OR z TUNE SHIFT. THE MAXIMUM CURRENT IN THE MACHINE DEPENDS OF WHICH OF x OR z CAN HAVE MAXIMUM TUNE SHIFT

MAXIMUM CURRENT DEPENDS ON ENERGY

EMITTANCE ALSO ENERGY DEPENDENT

NORMALIZED EMITTANCE

$$\varepsilon_{x,z} = \gamma^2 \tilde{\varepsilon}_{x,z}$$

$$\gamma = \frac{E}{E_0} \qquad E_0 = m_0 c^2$$

$$I_{MAX} = \frac{8\pi^2 b\, m_0 f_{REV}\, \gamma^3 \tilde{\varepsilon}_{x_0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)}{\mu_0 e (1+k)\sqrt{\beta_z^*}} \Delta\nu_{MAX}$$

MAXIMUM LUMINOSITY IS WHEN BOTH BEAMS ARE AT SPACE-CHARGE TUNE SHIFT LIMIT

$$\mathcal{L}_{MAX} = \frac{16\pi^3 b\, f_{REV}\, m_0^2 \gamma^4 \tilde{\varepsilon}_{x_0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right) \Delta\nu_{MAX}}{\mu_0^2 e^4 (1+k)\sqrt{\beta_x^*}\, \beta_z^{*\,3/2}}$$

MAXIMUM LUMINOSITY $\beta_z \to 0$

$\varepsilon_{x_0} \to$ BEAM PIPE APERTURE