Time Reversal ($t \to -t$)

A wavefunction cannot be in an eigenstate of *T*, because *T* is an antiunitary operator which changes functions to their complex conjugate. e.g. if

$$\psi = e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}$$

then (using $T\mathbf{p} = -\mathbf{p}$)

$$T\psi = e^{i(-\mathbf{p}\cdot\mathbf{x} + Et)}$$

$$= \psi^*$$

One can easily see that the eigenvalue equation

$$T\psi = \lambda\psi$$

has no solutions for any complex eigenvalue $\lambda = \rho e^{i\theta}$.

This does not mean T may not be a symmetry of the Hamiltonian. It just means the wavefunctions cannot be eigenstates.

Probabilities and expectation values are unchanged if T is a symmetry. e.g. For a free particle wave function

$$T(\psi^*\psi) = \psi\, \psi^* = \psi^*\psi$$

This may not be true for interactions if complex matrices are involved. T symmetry, as well as CP, is violated at the 0.2% level in neutral kaon decays.

Detailed Balance

Time reversal can be tested in several ways, for example, consider a two body reaction involving initial state spinless particles a & b, and final state spinless particles c & d:

$$a(\mathbf{p}_a) + b(\mathbf{p}_b) \rightarrow c(\mathbf{p}_c) + d(\mathbf{p}_d)$$

Under time reversal we get

$$c(-\mathbf{p}_c) + d(-\mathbf{p}_d) \rightarrow a(-\mathbf{p}_a) + b(-\mathbf{p}_b)$$

and applying parity we have

$$c(\mathbf{p}_c) + d(\mathbf{p}_d) \rightarrow a(\mathbf{p}_a) + b(\mathbf{p}_b)$$

So PT symmetry would imply that the rate for the reaction $a+b \rightarrow c+d$ should be the same as the rate for the reaction $c+d \rightarrow a+b$. This is the principal of detailed balance.

Antiparticles

Antiparticles exist even if charge conjugation is not a good symmetry.

- C-invariance says that changing particles into antiparticles should not change the physics
- The existance of antiparticles is a consequence of quantum mechanics and Lorentz invariance.

Free particle wave equations are based on kinematic laws, using the quantum mechanical substitutions

$$\vec{p} \rightarrow -i\hbar\vec{\nabla} \quad \text{and} \quad E \rightarrow i\hbar\frac{\partial}{\partial t}$$

giving the free particle wave equations, for example,

- non-relativistic Schrödinger equation

$$E = \frac{p^2}{2m} \quad \rightarrow \quad \frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi$$

- relativistic Klein-Gordon equation

$$E^2 = p^2c^2 + m^2c^4 \quad \rightarrow \quad \frac{\partial^2\psi}{\partial t^2} = \hbar^2c^2\nabla^2\psi - \frac{m^2c^4}{\hbar^2}\psi$$

relativistically

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

so relativistic quantum wave equations (e.g. Klein-Gordon or Dirac equations) have positive and negative energy solutions.

The negative energy solutions are antiparticles; antiparticles must exist if quantum mechanics and special relativity are correct.

- Particles and their antiparticles must have equal masses and lifetimes. This turns out to correspond to invariance under the combined CPT transformation.
- Particles and their antiparticles must have opposite “charges”, i.e. electric charge, baryon number, lepton number, ….
- If a particle carries no charges, then it is its own antiparticle. e.g. γ, π^0, ρ^0.
- Particles and their antiparticles do not have to interact the same way unless C-invariance is true.

If C-invariance is true, and a particle is interacting with a force, then the negative energy solution acts as if it had the opposite charge from the particle. This actually follows from solving the wave equations with interactions (e.g. a potential), but it there are several interpretations: Dirac holes in negative energy sea, Feynman particles moving backwards in time, or even by simply considering

$$E = \int \vec{F} \cdot d\vec{x}$$

One can see that forces for the positive and negative solutions should have opposite signs. (You can run into trouble applying this to gravity, however, since particles and antiparticles should both fall under gravity.)

Symmetries and Conservation Laws

A system is said to have a symmetry if the laws of physics for that system are invariant under some transformation.

For example, if the laws of physics are independent of location in space, then if $\psi(x)$ is a solution of the Schrödinger equation

$$i\hbar \frac{d\psi}{dt} = H\psi \quad \& \quad -i\hbar \frac{d\psi^*}{dt} = \psi^* H$$

then $\psi\ (x+\delta)$ is also a solution, where δ is a displacement in x. Consider an infinitesmal transformation ($\delta \to 0$). (This is quite general, since space translation is a continuous operation, so any arbitrary finite translation, Δ, can be made in infinitesmal steps.)

$$\psi(x+\delta) = \psi(x) + \delta\frac{\partial\psi}{\partial x} + \frac{\delta^2}{2!}\frac{\partial^2\psi}{\partial x^2} + \ldots \xrightarrow[\delta\to 0]{} \psi(x) + \delta\frac{\partial\psi}{\partial x}$$

But remembering that the momentum operator is $p_x = i\hbar\frac{\partial}{\partial x}$

$$\psi(x+\delta) = \left(1 - i\frac{\delta}{\hbar}p_x\right)\psi(x)$$

This must also be a solution of the Schrödinger equation

$$i\hbar \frac{d\psi(x+\delta)}{dt} = H\psi(x+\delta)$$

$$i\hbar \frac{d\psi(x)}{dt} + \delta \frac{d(p_x \psi(x))}{dt} = H\psi(x) - H\frac{i\delta}{\hbar} p_x \psi(x)$$

$$\therefore \quad \hbar \frac{d(p_x \psi(x))}{dt} = -iHp_x \psi(x)$$

Something (e.g. F) is conserved if its expectation value is independent of time. Assume the operator F does not depend on t, then

$$\frac{d}{dt}\langle F\rangle = \frac{d}{dt}\int d^3x\psi * F\psi = \int d^3x \frac{d\psi *}{dt} F\psi + \int d^3x\psi * \frac{dF\psi}{dt} = 0$$

In this case, we have assumed space invariance and the momentum operator has appeared, so let's check

$$\frac{d}{dt}\langle p_x\rangle = \int d^3x\left[\frac{d\psi *}{dt} p_x\psi + \psi * \frac{d(p_x\psi)}{dt}\right]$$

$$= \int d^3x\left[\frac{\psi^* H}{-i\hbar} p_x\psi - \psi^* \frac{i}{\hbar} Hp_x\psi\right]$$

$$= \int d^3x \frac{i}{\hbar}\left[\psi^* Hp_x\psi - \psi^* H p_x\psi\right] = 0!$$

So the assumption that physics is invariant under space transformations requires that momentum be invariant!

In general, it turns out that if a operator commutes with the Hamiltonian, the corresponding observable is a conserved quantity.

e.g. Energy, momentum, and angular momentum conservation correspond to invariance under time and space translations, and space rotations. Since these translations can be described by the Lorentz group, we can say that energy-momentum conservation are due to the Lorentz group symmetry of the universe.

Note, however, that we often write down Hamiltonians which do not conserve momentum or energy. *e.g.* When there is an external potential, or friction. In these cases momentum and energy may not be conserved, but this is because our wavefunction does not include the whole system.

Additive and Multiplicative Laws

If parity is conserved, then the parity of a system is the product of the parity of its components.

$$\begin{aligned}
\Psi(\vec{x}) &= \psi_1(\vec{x})\psi_2(\vec{x}) \\
P\Psi(\vec{x}) &= P(\psi_1(\vec{x})\psi_2(\vec{x})) \\
&= \psi_1(-\vec{x})\psi_2(-\vec{x}) \\
&= P_1\psi_1(\vec{x})P_2\psi_2(\vec{x}) \\
&= P_1P_2(\psi_1(\vec{x})\psi_2(\vec{x})) \\
&= P_\Psi\Psi(\vec{x})
\end{aligned}
\quad \Rightarrow \quad \boxed{P_\Psi = P_1P_2}$$

This is a general property of such discrete symmetries.

Continuous symmetries give additive conservation laws.

$$\begin{aligned}
\Psi(x+\delta) &= \psi_1(x+\delta)\psi_2(x+\delta) \\
&= \left(1 - i\frac{\delta}{\hbar}p_{x_1}\right)\psi_1(x)\left(1 - i\frac{\delta}{\hbar}p_{x_2}\right)\psi_2(x) \\
&= \left(1 - i\frac{\delta}{\hbar}p_{x_1} - i\frac{\delta}{\hbar}p_{x_2} - i\frac{\delta^2}{\hbar^2}p_{x_1}p_{x_2}\right)\psi_1(x)\psi_2(x) \quad \Rightarrow \quad \boxed{p_{x_\Psi} = p_{x_1} + p_{x2}} \\
&\xrightarrow[\delta \text{ small}]{} \left(1 - i\frac{\delta}{\hbar}(p_{x_1} + p_{x_2})\right)\Psi(x)
\end{aligned}$$