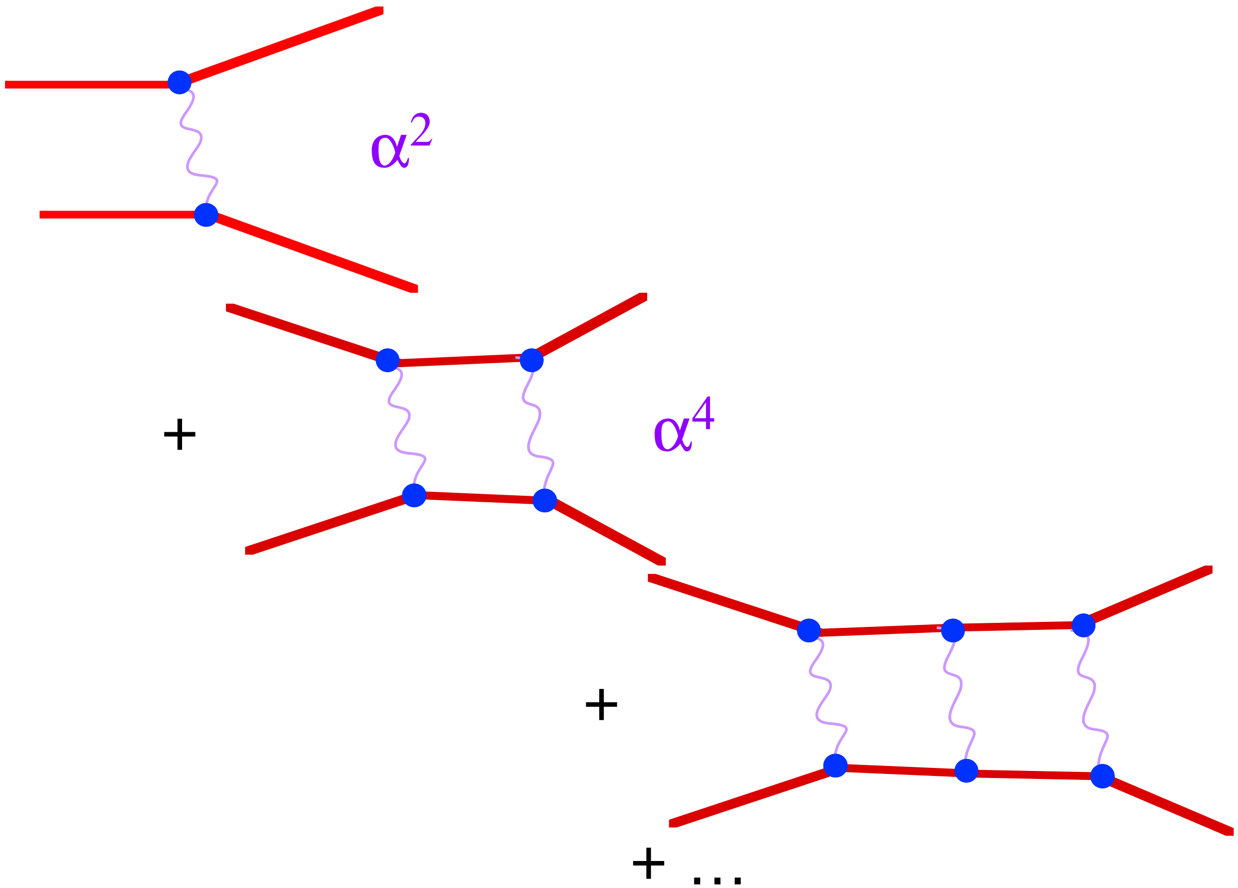
Perturbation Theory

If an interaction is weak (e.g. $\alpha << 1$), then the rate for a given processs can be calculated perturbatively, *e.g.* The total two body charged particle scattering rate is



Fermi's Golden Rule #2

Consider a system described by the time independent Schrödinger equation

$$i\hbar\frac{\partial\varphi}{\partial t} = H_0\varphi$$

with stationary state eigen solutions

$$\varphi = u_\alpha(\vec{x})e^{-iE_\alpha t/\hbar}$$

If there is a small interaction coupling different states, then $H=H_0+H_{int}$, and the lowest order transition rates can be calculated perturbatively.

For a direct transition from an initial state $|\alpha\rangle$ to a final state $|\beta\rangle$:

Fermi's Golden Rule No. 2

$$w_{\beta\alpha} = \frac{2\pi}{\hbar}\left|\langle\beta|H_{int}|\alpha\rangle\right|^2\rho(E)$$

where

$$\langle\beta|H_{int}|\alpha\rangle = \int d^3x \cdot u_\beta^*(\vec{x})\cdot H_{int}\cdot u_\alpha(\vec{x})$$

is the integral over space of the coupling of the space parts of the stationary states, and the density of states phase space is

$$\frac{dN}{dE} \equiv \rho(E)$$

i.e. The transition rate is the square of the amplitude for interaction coupling the two states together times the phase space for the transition.

Fermi's Golden Rule #1

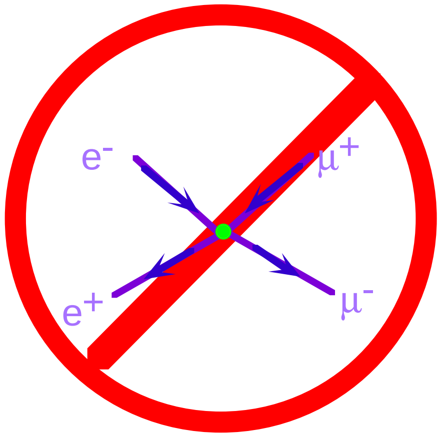
If there is no direct coupling between the initial and final states, the transition rate from an initial state |α⟩ to a final state |β⟩ via intermediate states |n⟩ is given by:

- Fermi's Golden Rule No. 1

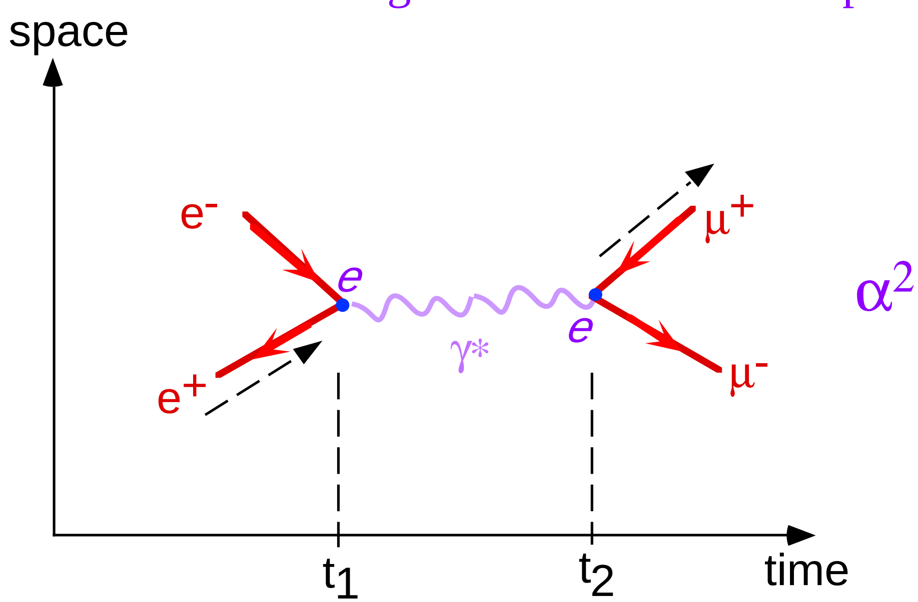
$$w_{\beta\alpha} = \frac{2\pi}{\hbar}\left| -\sum_{n\neq\alpha} \frac{\langle\beta|H_{\text{int}}|n\rangle\langle n|H_{\text{int}}|\alpha\rangle}{E_n - E_\alpha}\delta(E_\beta - E_\alpha)\right|^2 \rho(E)$$

$e^+e^- \to \mu^+\mu^-$

Consider the cross section for electron-positron annihilation into a muon pair via an intermediate virtual photon. There is no direct 4-fermion coupling between electrons and muons, i.e.

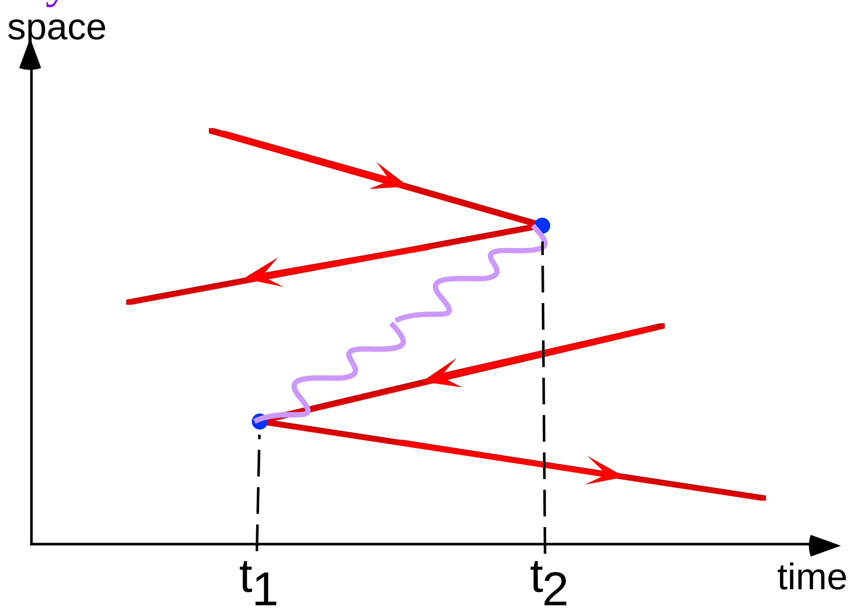


So the dominant mechanism is via a single intermediate virtual photon



Total Matrix Element

The total amplitude the sum of all possible time-ordered diagrams. In this case there is one additional possibility:



The total matrix element for $e^+e^- \to \mu^+ \mu^-$ is the sum of these two intermediate states, *i.e.*

$$M \sim \left\{ \frac{\langle \mu^+\mu^- | H_{em} | \gamma \rangle \langle \gamma | H_{em} | e^+e^- \rangle}{E_{e^+e^-} - E_\gamma} + \frac{\langle \mu^+\mu^- | H_{em} | \gamma \rangle \langle \gamma | H_{em} | e^+e^- \rangle}{E_{e^+e^-} - E_{e^+e^-} - E_{\mu^+\mu^-} - E_\gamma} \right\} \delta\left(E_{e^+e^-} - E_{\mu^+\mu^-} \right)$$

$$= \frac{\langle \mu^+\mu^- | H_{em} | \gamma \rangle \left(-E_{\mu^+\mu^-} + E_{e^+e^-} - 2E_\gamma \right) \langle \gamma | H_{em} | e^+e^- \rangle}{-E_{e^+e^-} E_{\mu^+\mu^-} + \left(E_{\mu^+\mu^-} - E_{e^+e^-} + E_\gamma \right) E_\gamma} \delta\left(E_{e^+e^-} - E_{\mu^+\mu^-} \right)$$

$$= \langle \mu^+\mu^- | H_{em} | \gamma \rangle \frac{2E_\gamma}{E^2_{e^+e^-} - E^2_\gamma} \langle \gamma | H_{em} | e^+e^- \rangle$$

where

$$E^2_{e^+e^-} = \left(p_{e^+} + p_{e^-} \right)^2 + \left(\vec{\mathbf{p}}_{e^+} + \vec{\mathbf{p}}_{e^-} \right)^2$$

$$E^2_\gamma = \vec{\mathbf{p}}^2_\gamma + m^2_\gamma$$

and

$$p_{e^-} = \left(\frac{E_{e^-}}{c}, \vec{p}_{e^-} \right)$$

is the four-momentum of the electron, and similarly for the positron.

$e^+e^- \rightarrow \mu^+ \mu^-$ Matrix Element

Using 3-momentum conservation

$$\vec{\mathbf{p}}_\gamma = \vec{\mathbf{p}}_{e^+} + \vec{\mathbf{p}}_{e^-}$$

we then have that

$$\frac{1}{E^2_{e^+e^-} - E^2_\gamma} = \frac{1}{\left(p_{e^+} + p_{e^-}\right)^2 - m^2_\gamma}$$

$$= \frac{1}{q^2}$$

where q^2 is the 4-momentum-squared of the the virtual photon.

The gamma-fermion coupling terms are

$$\left\langle \mu^+\mu^- \middle| H_{em} \middle| \gamma \right\rangle \cong e \qquad \textit{and} \qquad \left\langle \gamma \middle| H_{em} \middle| e^+e^- \right\rangle \cong e \approx \sqrt{\alpha}$$

So we have that the matrix element is of the form

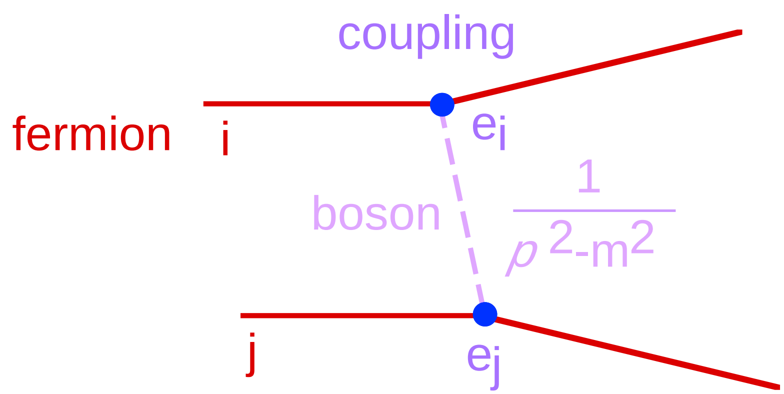
$$M \sim \frac{e^2}{q^2} \sim \frac{\alpha}{q^2}$$

Feynman Diagrams

In general, the exchange of a particle of mass m and 4-momentum p corresponds to a propagator term

$$\frac{1}{p^2 - m^2}$$

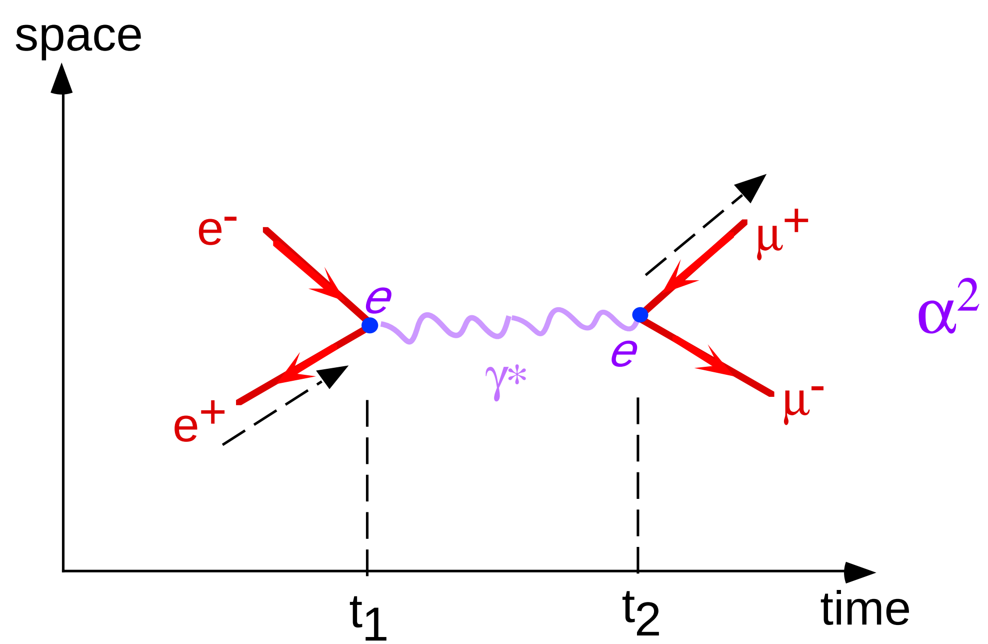
in the matrix element



Relativistic quantum field theory is beyond the scope of this course, the amplitude of a process is calculated by drawing all the Feynman diagrams, writing down for each diagram the product of the vertex factors (couplings), the internal propagators due to exchanged particles, and factors corresponding to the polarization of the incoming and outgoing particles, and summing over all diagrams. The the total rate (e.g. cross section) is the square of the amplitude times the appropriate density of states factor and the cross section is of the form

$$\boxed{\sigma\left(e^+e^- \to \mu^+\mu^-\right) \sim M^2 \sim \frac{\alpha^2}{q^4}}$$

Electron-Positron Annihilation



By counting the vertex and propagator factors, we see that the matrix element for electron-positron annihilation into muon pairs is of the form:

$$M \sim \frac{e^2}{q^2}$$

where q^2 is the four-momentum of the virtual photon.

$e^+e^- \to \mu^+ \mu^-$ Total Cross Section

The cross section is thus of the form

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) \sim M^2 \sim \frac{\alpha^2}{q^4}$$

But this is dimensionally incorrect, so the phase space and other factors must have units of $(\text{energy})^2$.

In the high energy limit, where the masses of all the fermions can be neglected, the only relevant variable with units of energy is q^2. So the cross-section must be of order:

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) \sim \frac{\alpha^2}{q^2}$$

The correct total cross section for the annihilation of point electrons into point muon pairs via electromagnetic interactions is (in lowest order)

$$\boxed{\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi}{3}\frac{\alpha^2}{s}}$$

where $s=q^2$ is the square of the total c.m. energy of the electron-positron annihilation. In the c.m. frame, $s=(2E_e{}^2)$.