# Wave Optical Model for Scattering

Consider a beam of spinless particles travelling in the +z direction:

$$\psi_i = e^{i(kz-\omega t)}$$

(This wave has unit amplitude; $k=2\pi/\lambda=2\pi p/h$, $\lambda$=deBroglie wavelength defined in c.m frame.) For $kr$>>1 this plane wave can be expanded into a sum of spherical waves (of angular momentum $l$):

$$\psi_i = \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1) \left\{ \underbrace{(-1)^l e^{-ikr}}_{\text{incoming}} - \underbrace{e^{ikr}}_{\text{outgoing}} \right\} P_l(\cos\theta)\, e^{-i\omega t}$$

If the plane wave is incident on a spinless target particle it can scatter, but by causality the scatterer cannot affect the incoming wave, only the outgoing wave, so

$$\psi_{total} = \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1) \left\{ (-1)^l e^{-ikr} - \eta_l e^{i2\delta_l} e^{ikr} \right\} P_l(\cos\theta)\, e^{-i\omega t}$$

where

- $2\,\delta_l$ is the phase change of the $l^{th}$ component of the wave.
- $\eta_l$ is the amplitude ($0 < \eta_l < 1$) for elastic scattering (*i.e.* with no change in kinetic energy).
- Legendre Polynomials $P_l(\cos\theta) \equiv \frac{1}{2^l\, l!} \left( \frac{d}{d(\cos\theta)} \right)^l \left( -\sin^2\theta \right)^l$

# The Scattered Wave

The scattered wave, $\psi_{total}$-$\psi_i$, is

$$\psi_{scatt} = \frac{e^{i(kr-\omega t)}}{r}\underbrace{\left[\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\frac{\left(\eta_l e^{2i\delta_l}-1\right)}{2i}P_l(\cos\theta)\right]}_{\substack{\parallel \\ f_l}}$$

$$= \frac{e^{i(kr-\omega t)}}{r}F(\theta)$$

The rate of scattering into a solid angle dΩ is

$$df \equiv v_{out}\left(\psi_{scatt}\psi^*_{scatt}\right)r^2 d\Omega = v_{out}\left|F(\theta)\right|^2 d\Omega$$

The rate of scattering into a solid angle dΩ is also, by definition,

$$df \equiv \frac{d\sigma}{d\Omega}d\Omega\ \ \phi_{in} = \frac{d\sigma}{d\Omega}d\Omega\ \ n\ \ v_{in}$$

$$\therefore \frac{d\sigma}{d\Omega}\ n\ \ v_{in} = v_{out}\left|F(\theta)\right|^2$$

The incident flux is

$$\phi_{in} \;=\; n\ v_{in} \;=\; \psi_{scatt}\psi^*_{scatt}\ \ v_{in}$$

Since $v_{in} = v_{out}$ for elastic scattering

$$\frac{d\sigma}{d\Omega}\ n\ = \left|F(\theta)\right|^2$$

# Elastic Cross Section

Normalizing to one particle ($n$=1), we have the differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{el} = \left|F(\theta)\right|^2 = \frac{1}{k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)\left|\frac{\left(\eta_l e^{2i\delta_l}-1\right)}{2i}\right|\left|\frac{\left(\eta_{l'} e^{2i\delta_{l'}}-1\right)}{2i}\right| P_l(\cos\theta)P_{l'}(\cos\theta)$$

Integrating over $d\Omega$, using the normalization property of Legendre Polynomials

$$\int_{4\pi} P_l(\cos\theta)P_{l'}(\cos\theta)d\Omega = \frac{4\pi\delta_{ll'}}{2l+1},$$

Normalizing to one particle ($n$=1), we have the differential cross section

$$\sigma_{el} = \int_{4\pi}\left(\frac{d\sigma}{d\Omega}\right)_{el} d\Omega = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left|\frac{\left(\eta_l e^{2i\delta_l}-1\right)}{2i}\right|^2$$

For no absorption, $\eta$=1, we have pure elastic scattering

$$\sigma_{el} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

$$= 0 \qquad\qquad for \;\; \delta_l = 0$$

# Total Cross Section

Whatever scattering isn't elastic must be inelastic:

$$\sigma_{scatt} = \sigma_{el} + \sigma_{inel}$$

and by **conservation of probability**

$$\sigma_{inel} = \int \left( \left|\psi_{in}\right|^2 - \left|\psi_{out}^{el}\right|^2 \right) r^2 d\Omega = \left[ \int \psi_{in}^* \psi_{in} r^2 d\Omega \right] - \sigma_{el}$$

where

$$\psi_{in} = \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1)(-1)^l e^{-ikr} P_l(\cos\theta)$$

$$\therefore \sigma_{inel} = \pi \lambda\!\!\!^{-2} \sum_{l=0}^{\infty} (2l+1)\left(1 - \eta_l^2\right)$$

$$\therefore \sigma_{total} = \sigma_{el} + \sigma_{inel} = \pi \lambda\!\!\!^{-2} \sum_{l=0}^{\infty} (2l+1) 2 (1 - \eta_l \cos 2\delta_l)$$

Since probabilities must be between 0 and 1, *i.e.* $0<\eta_l<1$ (the **Unitarity** condition)

$$\boxed{\sigma_{total} = 4\pi \lambda\!\!\!^{-2} \sum_l (2l+1) = \frac{16\pi}{s} \sum_l (2l+1)}$$

This is an absolute maximum cross-section. (*i.e.* In general, there is an upper limit of $4\pi\lambda\!\!\!^{-2}$ for each total angular momentum state.)

# Unitarity

The cross section for any distinguishable scattering process has a cross section less than

$$\sigma_{max} = 4\pi/p_{cm}^2 = 16\pi/s = (19.6 \text{ mb GeV}^2)/s \qquad \text{(in high energy limit)}$$

Examples:

(1) $e^+e^-$ annihilation into point fermion-antifermion pair

$$\sigma\left(e^+e^- \to f\bar{f}\right) = \frac{4\pi}{3}\frac{\alpha^2}{s}Q_f^2$$

$$= \frac{\alpha^2}{12}Q_f^2\sigma_{max} = 4.4\times10^{-6}Q_f^2\sigma_{max}$$

No problem for muons ($Q_\mu$=1) or quarks ($Q_u$= $Q_c$= $Q_t$=2/3, $Q_d$= $Q_s$= $Q_b$=1/3), but what if a point fermion existed with charge $Q_f$=475?

**Answer**: Perturbation theory would break down, and the QED cross section would be reduced below the unitarity bound.

(2) billiard balls (or neutron-proton scattering at high energies) have a essentially constant total cross section

$$\sigma_{billiards} = 4\pi R^2$$

This violates unitarity if $p_{cm}$>2/R!

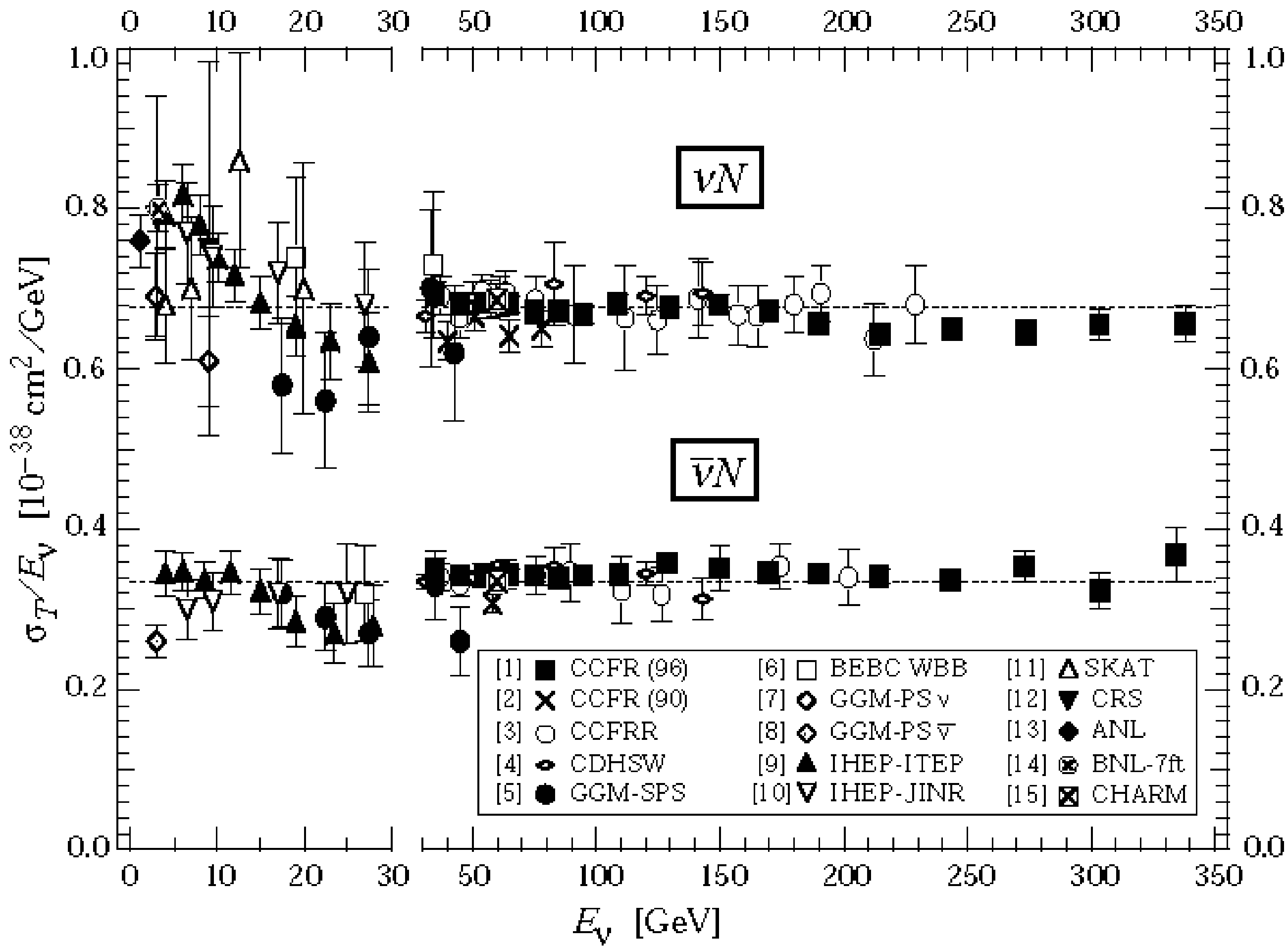
**Answer**:: In billiard ball scattering, the number of orbital angular momentum states is proportional to *s*, so the constant cross section is maintained because the number of scattering states increases at a rate which cancels out the reduction in the maximum cross section per scattering state.

# The universe never fails, but our theories do!

(3) neutrino-proton interactions

$$\sigma(\nu_\mu p) = 0.4\text{x}10^{-38}\text{cm}^2/\text{GeV}^2\ s = \sigma_{max}\ s^2/(150\ \text{GeV})^4$$

This cross section formula cannot be true for c.o.m. energies above 150 GeV!




http://pdg.lbl.gov/2001/hadronicrpp_page6.pdf

# Unitarity ≈10G$

(3) neutrino-proton interactions (cont'd)

$$\sigma(\nu_\mu p) = 0.4\times10^{-38}\text{cm}^2/\text{GeV}^2 = \sigma_{max}\ s^2/(150\ \text{GeV})^4$$

This cross section formula cannot be true for c.o.m. energies above 150 GeV!

A Yukawa force mediated by a massive particle

$$V(x) = (-g^2/x)\ \exp(-xM_B)$$

gives a cross section

$$\sigma = 4\pi g^4 s/M_B{}^4$$

in the low energy limit. In the high energy limit (corresponding to x very small) this potential is just a colomb-like potential and the cross section turns over and falls as 1/s just like the $e^+e^-$ annihilation cross-section. The boson must have a mass less than 150 GeV (Prediction!), so that the turn-over occurs before the unitarity violation would occur.

(4) WW scattering violates unitarity at about 1.5 TeV

Theoretical solution: Higgs or supersymmetry or W substructure or …

Experimental solution: Build the biggest accelerator you can convince your government(s) to afford (SSC - nope, LHC yes), and see what happens.