University of Toronto PHY357S

Problem Set 3 Answers

due Wednesday, 20 March 2002

(Late penalty is 10% per day, and no problem set is accepted after March 22.)

Each problem is of equal weight, but **not all problems may be marked**.

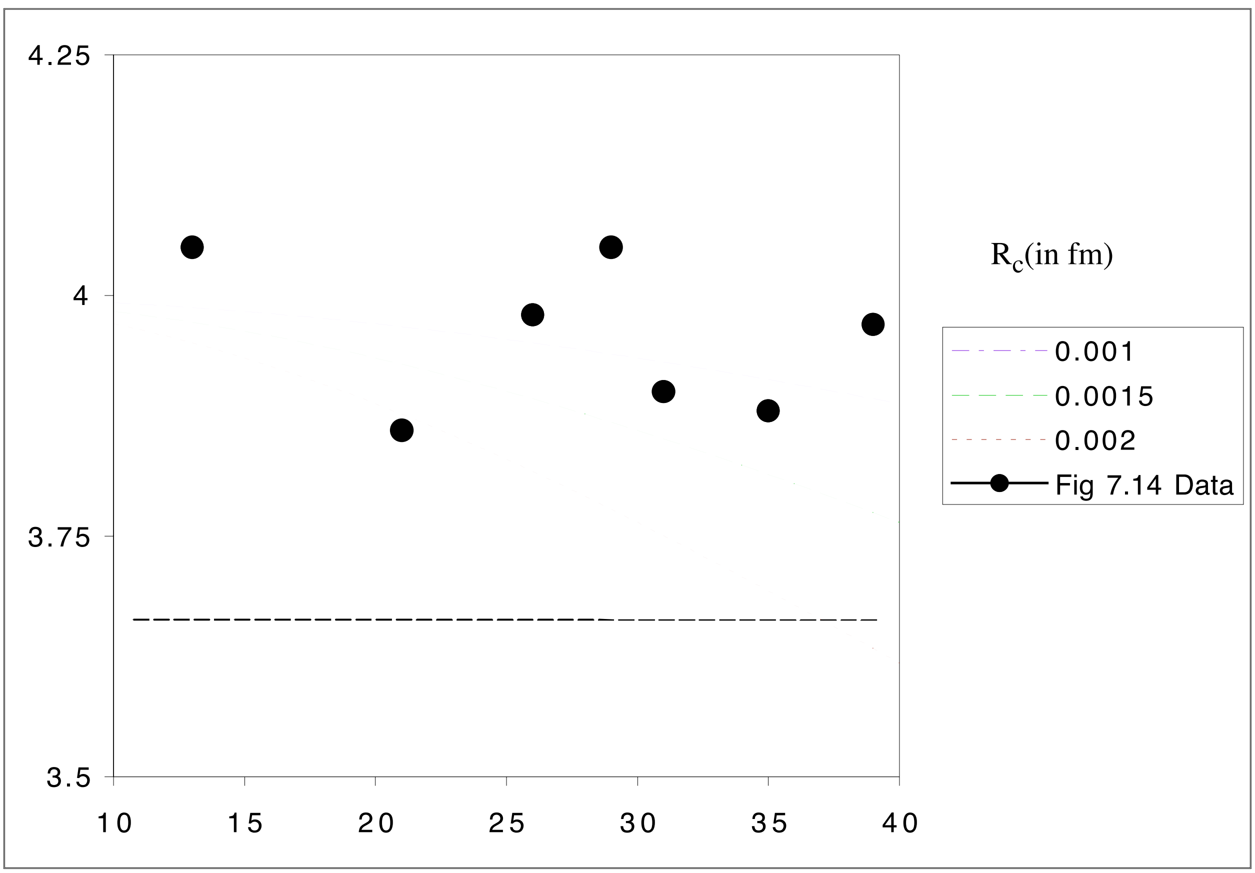
These problems are primarily based on Chapters 1-9 of Martin and Shaw and on the lectures up to March 11; Appendices A, B and E and the inside back cover of M&S may also be helpful.

1) Use the data shown in Martin & Shaw Figure 7.14 to set an upper limit on the charge radius of charm quarks. (The solid line in the figure is the prediction assuming all quarks are point fermions.) Any reasonable criterion for setting the upper limit will be accepted. Assume a dipole form factor for the size of the quarks.

From the lectures we have a dipole form $\sigma(e^+e^- \to \mu^+\mu^-) = (4\pi\alpha^2/3s)/(1+sR^2)^2$, (a factor $\sim 1/(1+sR^2)^4$ also makes sense and was accepted), so for charm quarks we expect

$$\sigma\left(e^+e^- \to c\bar{c}\right) = 3\frac{4\pi\alpha^2 q_c^2}{3s\left(1+sR_c^2\right)^2} = \frac{16\pi\alpha^2}{3s\left(1+sR_c^2\right)}$$

Charm quarks contribute about (4/3)/(11/3)=4/11=36% to the R ratio shown in the Figure 7.14. If charm quarks have a size, when s is greater than the size the production cross section will drop dramatically. Using the above formula, we can calculate what the R ratio should look like if only the charm quark has a size, and the other quarks are point particles. Plotting the data from the figure, we can easily see that is consistent with the data, but R_c=0.002 is not. (To save



time I did not plot the error bars from Fig. 7.14, but I do keep them in mind.) So the upper limit on the size of a charm quark is about 0.002fm.

2) Assume that the momentum distributions of valence u quarks in the proton and valence $\bar{d}$ quarks in the antiproton have the forms:

$$F_u(x) = x\,u(x) = a_1(1-x)^3,$$

$$F_{\bar{d}}(x) = x\,\bar{d}(x) = a_2(1-x)^3,$$

where x is the Bjorken *scaling variable* (fractional momentum of nucleon carried by a quark.

a) If the quarks account in total for half the nucleon momentum, find a_1 and a_2. (Neglect the existence of sea quarks; gluons carry the other half of the nucleon momentum.)

A proton has valence quark content *uud*, so if we ignore sea quarks then by isospin invariance the *u* quarks must carry twice as much momentum as the *d* quarks in a proton, *i.e.* R_c (in fm)

$$\int_0^1 x\mathrm{F}_u(x)dx = \int_0^1 x\mathrm{F}_d(x)dx$$

Looking at the assumed momentum distribtions, this can only be true if

$$a_1 = 2a_2$$

If quarks carry half the proton momentum

$$\int_0^1 x\mathrm{F}_u(x)dx + \int_0^1 x\mathrm{F}_d(x)dx = \frac{1}{2}$$

$$\therefore \int_0^1 (a_1 + a_2)(1-x)^3\,dx = \frac{1}{2}$$

$$\therefore 3a_2\left(\frac{1}{4}\right) = \frac{1}{2}$$

$$\therefore a_2 = \frac{2}{3} \qquad \text{and} \qquad a_1 = \frac{4}{3}$$

(For more background, see M&S Problem 7.7, and M&S sections 7.4.3 and 7.5.1.)

b) Write down the resonant cross section (as a function of $\hat{s}$) $\sigma_{u\bar{d}} \rightarrow W^+(\hat{s})$ for $u\bar{d} \rightarrow W^+$ production, where $\hat{s}$ is the c.o.m. energy of the $u\bar{d}$ collision. Assume the quarks and antiquarks are inside a proton or antiproton. *Remember that quarks come in different colours and the W boson is colourless*.

The resonant cross section (from Lecture 17 or M&S Section B.5)is

$$\sigma\left(u\bar{d} \rightarrow W^+\right) = \frac{4\pi\lambda\!\!^-{}^2(2J_W+1)\Gamma_W\Gamma_{W\rightarrow u\bar{d}}/4}{(2s_u+1)(2s_{\bar{d}}+1)\left[\left(\sqrt{\hat{s}} - M_W\right)^2 + \Gamma_W^2/4\right]}$$

$$= \frac{\pi\lambda\!\!^-{}^2(2\cdot 1+1)\Gamma_W\Gamma_{W\rightarrow u\bar{d}}}{\left(2\cdot\frac{1}{2}+1\right)\left(2\cdot\frac{1}{2}+1\right)\left[\left(\sqrt{\hat{s}} - M_W\right)^2 + \Gamma_W^2/4\right]}$$

$$= \frac{3\pi\lambda\!\!^-{}^2\Gamma_W\Gamma_{W\rightarrow u\bar{d}}}{\left[4\left(\sqrt{\hat{s}} - M_W\right)^2 + \Gamma_W^2\right]}$$

But quarks (and antiquarks) come in 3 colours, but the W is colourless, so 2/3 of the time the colour of the quark and will not be the anticolour of the antiquark, *i.e.* the cross must be reduced by a factor of 3 to take into account of colour conservation

$$\sigma\left(u\bar{d} \to W^+\right) = \frac{\pi\lambda\!\!\!^{-2}\Gamma_W\Gamma_{W\to u\bar{d}}}{\left[4\left(\sqrt{s} - M_W\right)^2 + \Gamma_W^2\right]}$$

Note: One should also take into account of the fact that only one helicity state of the spin-1 W actually contributes (*i.e.* the W must be left handed), which reduces the cross section by another factor of 3, but this goes beyond what we have done.

c) By considering all possible decays of the W boson (*e.g.* Martin & Shaw Figs. 8.10 & 8.20), estimate the branching ratio for $W^+ \to u\,\bar{d}$.

There are 9 possible W^+ decays:

$$\begin{aligned} W^+ &\to \nu_e e^+ \\ &\to \nu_\mu \mu^+ \\ &\to \nu_\tau \tau^+ \\ &\to u\bar{d} \;\times\; 3 \text{ colour - anticolour possibilities} \\ &\to c\bar{s} \;\times\; 3 \text{ colour - anticolour possibilities} \end{aligned}$$

One third of these are $W^+ \to u\,\bar{d}$, *i.e.* the branching ratio is

$$BR(W^+ \to u\bar{d}) = \frac{3}{1+1+1+3+3} = \frac{1}{3}$$

d) What is the value, σ_{peak}, of the $u\bar{d} \to W$ cross-section at the peak of the resonance?

The resonance at the peak is just

$$\sigma_{peak}\left(u\bar{d} \to W^+\right) = \frac{\pi\lambda\!\!\!^{-2}\Gamma_W\Gamma_{W\to u\bar{d}}}{\left[4\left(M_W - M_W\right)^2 + \Gamma_W^2\right]} = \frac{\pi\lambda\!\!\!^{-2}\Gamma_{W\to u\bar{d}}}{\Gamma_W} = \pi\lambda\!\!\!^{-2} BR\left(W \to u\bar{d}\right) = \frac{4\pi}{M_W^2} BR\left(W \to u\bar{d}\right)$$

$$\begin{aligned} \sigma_{peak}\left(u\bar{d} \to W^+\right) &= \frac{4\pi}{M_W^2} BR\left(W \to u\bar{d}\right) \\ &= \frac{4\pi}{(80.33\text{GeV})^2}\frac{1}{3} \times \left(\hbar c = 1.973 \times 10^{-16}\text{GeV} \cdot \text{fm}\right)^2 \\ &= 2.5 \times 10^{-35} m^2 = 0.25\mu\text{b} \end{aligned}$$

e) Make a "narrow width" approximation that the $u\bar{d} \to W$ resonance is a delta function. (*i.e. For a resonance of mass M and width Γ,* $\sigma(s) \sim \delta\left(1 - s/M^2\right)\sigma_{peak}\,\Gamma/M$.) Integrate the cross-section over the quark distributions of part (a) to calculate the cross-section for $p\bar{p} \to W^+ +$ *anything*. *(This part has some very tedious integration. Don't forget that* $\int_{x-\varepsilon}^{x+\varepsilon}\delta(x)dx = 1$.*)*

In the narrow width approximation,

$$\sigma(s) \sim \delta\left(1 - \frac{s}{M_W^2}\right)\sigma_{peak}\left(u\bar{d} \to W^+\right)\frac{\Gamma_W}{M_W}$$

The total cross section in this case is calculated by integrating the probability of have a quark having a certain momentum times the probability of an antiquark having a certain momentum times the cross section for W production at the s value resulting from that combination of quark and antiquark momenta. (See M&S Section 7.4.2 for the probabilistic definition of the momentum distributions.) *i.e.*

$$\sigma_{p\bar{p}\to W^+}(s) = \int_0^1\int_0^1 \sigma_{u\bar{d}}(\hat{s})\cdot u(x_u)\cdot \bar{d}(x_{\bar{d}})\cdot dx_u\cdot dx_{\bar{d}}$$

The structure of the proton is determined by the strong interaction, so by C invariance, the up quark content $F_u(x)$ in a proton must be the same as the anti-up content of the antiproton, $F_{\bar{u}}(x)$, and the down quark distribution in the proton is the same as the anti-down distribution in the antiproton.

$$\therefore \sigma_{p\bar{p}\to W^+}(s) = \int_0^1\int_0^1 \sigma_{u\bar{d}}(\hat{s})\cdot u(x_u)\cdot d(x_{\bar{d}})\cdot dx_u\cdot dx_{\bar{d}}$$

Using the momentum distributions given above and the narrow width cross section approximation we have

$$\therefore \sigma_{p\bar{p}\to W^+}(s) \sim \int_0^1\int_0^1 \delta\left(1-\frac{\hat{s}}{M_W^2}\right)\sigma_{peak}\left(u\bar{d}\to W^+\right)\frac{\Gamma_W}{M_W}\cdot a_1\frac{(1-x_u)^3}{x_u}\cdot a_2\frac{(1-x_{\bar{d}})^3}{x_{\bar{d}}}\cdot dx_u\cdot dx_{\bar{d}}$$

The quark-antiquark centre-of-mass energy squared is just $\hat{s} = x_u x_{\bar{d}} s$, so

$$\sigma_{p\bar{p}\to W^+}(s) \sim \sigma_{peak}\left(u\bar{d}\to W^+\right)\frac{\Gamma_W}{M_W}a_1a_2\int_0^1\int_0^1 \delta\left(1-\frac{x_u x_{\bar{d}} s}{M_W^2}\right)\cdot\frac{(1-x_u)^3}{x_u}\cdot\frac{(1-x_{\bar{d}})^3}{x_{\bar{d}}}\cdot dx_u\cdot dx_{\bar{d}}$$

$$= \sigma_{peak}\frac{\Gamma_W}{M_W}\frac{8}{3}\int_0^1\int_0^1 \delta\left(1-\frac{x_u s}{M_W^2}x_{\bar{d}}\right)\cdot\frac{(1-x_u)^3}{x_u}\cdot\frac{(1-x_{\bar{d}})^3}{x_{\bar{d}}}\cdot dx_u\cdot dx_{\bar{d}}$$

$$= \frac{8}{3}\sigma_{peak}\frac{\Gamma_W}{M_W}\int_{\frac{M_W^2}{s}}^1 \frac{(1-x_u)^3}{x_u}\cdot\frac{\left(1-\frac{M_W^2}{x_u s}\right)^3}{\frac{M_W^2}{x_u s}}\cdot dx_u\,\frac{M_W^2}{x_u s}$$

where the change in the lower bound of integration is just because we must have $\frac{M_W^2}{s} < x_u < 1$, since if $x_u < \frac{M_W^2}{s}$, then $\hat{s} < M_W^2$. So, defining, $k \equiv \frac{M_W^2}{s}$, then

$$\sigma_{p\bar{p}\to W^+}(s) \sim \frac{8}{3}\sigma_{peak}\frac{\Gamma_W}{M_W}\int_k^1 \frac{(1-x_u)^3}{x_u}\cdot\left(1-\frac{k}{x_u}\right)^3\cdot dx_u$$

After some straightforward but very, very tedious integration one gets (I hope, since I didn't use Maple or Mathematica)

$$\sigma_{p\bar{p}\to W^+}(s) \sim \frac{8}{3}\sigma_{peak}\frac{\Gamma_W}{M_W}\left[-\left(1+9k+9k^2+k^3\right)\ln k+\tfrac{11}{3}-9k+9k^2+\tfrac{11}{3}k^2\right]$$

$$= \frac{8}{3}(0.25\mu b)\frac{2.1GeV}{80.3GeV}\left[-\left(1+9k+9k^2+k^3\right)\ln k+\tfrac{11}{3}-9k+9k^2+\tfrac{11}{3}k^2\right]$$

$$= 18nb\left[-\left(1+9k+9k^2+k^3\right)\ln k+\tfrac{11}{3}-9k+9k^2+\tfrac{11}{3}k^2\right]$$

f) Estimate the total cross sections for $\sigma_{p\bar{p}} \to W^+$ at

i) the UA1 experiment at CERN ($\sqrt{s}$ =0.54 TeV in 1982),

k=(80.3GeV/540GeV)^2, so

$$\sigma_{p\bar{p}\to W^+} \sim 140nb$$

The actual cross section is about 5nb. (When I set up the problem I made a mistake and got an answer of 14nb, which I thought wasn't bad. Clearly I should have used softer quark momentum distributions, *e.g.* ~(1-x)^5.

ii) the CDF experiment at Fermilab ($\sqrt{s}$ =2TeV in 2002), and

k=(80.3GeV/2000GeV)^2, so

$$\sigma_{p\bar{p}\to W^+} \sim 180nb$$

The actual cross section is about 25nb. The cross section increases because more partons have enough energy to make W bosons.

3) There is very weak evidence from CERN for a boson with a mass around 115GeV.

a) Such a Higgs would be similar in mass to the Z^0. Estimate the branching ratios of such a Higgs into each possible type of quark-antiquark or lepton-antilepton pair, and compare these to the branching ratios of the Z^0. *i.e.* Write down a table with the branching ratios into each possible pair for both the Higgs and the Z^0.

This is a slightly higher energy and expanded version of M&S Problem 9.6, but since the higher mass doesn't open up any new decay channels (except some radiative decays into real W and Z bosons which we will ignore), the branching ratios are exactly the same as given in the "Hints" given in M&S for Problem 9.6. *i.e.* $BR(\mathrm{H}^0 \to \mathrm{f\bar{f}}) \propto m_f^2 \times (colour \;\; factor)$:

$$\sum_f BR(\mathrm{H}^0 \to \mathrm{f\bar{f}}) \propto 3m_b^2 + 3m_c^2 + 3m_s^2 + 3m_d^2 + 3m_u^2 + m_\tau^2 + m_\mu^2 + m_e^2 = 69 \;\; to \;\; 72GeV^2$$

The only real puzzle here is what mass to use for the light quarks. M&S give the *u, d*, *s* and *c* quark masses to be about 350, 350, 500, and 1500 MeV, while in the lectures I describe them as having masses of 5, 10, 200, and 1300 MeV. The difference is that the "constituent mass" quoted by M&S includes the binding energy of the QCD gluon fields associated with the quark when it is inside a hadron. Most of the mass of a light hadron is in this glue. The mass I quote is the so called "current mass" or "bare mass". It is okay to use either constituent or bare masses in this problem, since this is a subtlety we have not discussed in detail. I use both and give the range in my answers (the first number uses bare masses, the second constituent masses)

$$BR(\mathrm{H}^0 \to \mathrm{f\bar{f}}) = \frac{m_f^2 \times (colour \quad factor)}{3m_b^2 + m_\tau^2 + 3m_c^2 + 3m_s^2 + m_\mu^2 + 3m_d^2 + 3m_u^2 + m_e^2}$$

$$\therefore \qquad BR(\mathrm{H}^0 \to \mathrm{b\bar{b}}) = 88\% \;\; to \;\; 84\%$$

$$BR(\mathrm{H}^0 \to \mathrm{c\bar{c}}) = 7.3\% \;\; to \;\; 9.4\%$$

$$BR(\mathrm{H}^0 \to \mathrm{s}\bar{s}) = 0.2\% \;\; to \;\; 1\%$$

$$BR(\mathrm{H}^0 \to d\bar{d}) = 4 \times 10^{-6} \;\; to \;\; 0.5\%$$

$$BR(\mathrm{H}^0 \to u\bar{u}) = 1 \times 10^{-6} \;\; to \;\; 0.5\%$$

$$BR(\mathrm{H}^0 \to \tau^+\tau^-) = 4.6\% \;\; to \;\; 4.4\%$$

$$BR(\mathrm{H}^0 \to \mu^+\mu^-) = 1.6 \times 10^{-4} \;\; to \;\; 1.5 \times 10^{-4}$$

$$BR(\mathrm{H}^0 \to e^+e^-) = 4 \times 10^{-9}$$

It would clearly be an interesting and very challenging measurement (if the higgs is discovered) the measure the branching ratios into light quarks.

b) Use dimensional arguments to make a rough estimate of the total decay width of a 115 GeV Higgs boson and compare it to the total decay width of the Z^0.

This answer is just the same as for M&S Problem 9.6, *i.e.* Γ_{total}~O(10MeV).