PARTICLE FLUX IN NON RELATIVISTIC QM

FREE PARTICLE

$$E = \frac{p^2}{2m}$$

$$E \to i\hbar \frac{\partial}{\partial t} \qquad \bar{p} \to -i\hbar \bar{\nabla} \qquad (\hbar \to 1)$$

SCHRÖDINGER $\quad i\frac{\partial \psi}{\partial t} + \frac{1}{2m}\nabla\psi = 0$

INTERPRET $\quad \rho = |\psi|^2$ PROBABILITY DENSITY

FOR MOVING PARTICLES, INTERESTED IN DENSITY FLUX $\bar{J}$ ← CURRENT

- RATE OF DECREASE IN VOLUME = TOTAL FLUX OUT OF VOLUME

$$-\frac{\partial}{\partial t}\int_V \rho \, dV = \underbrace{\int_S \bar{j}\cdot\bar{n}\, ds = \int_V \bar{\nabla}\cdot\bar{j}\, dv}_{\text{GAUSS}}$$

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot\bar{j} = 0$$

CONTINUITY EQUATION

WHAT ABOUT FLUX IN SCHRÖDINGER ?

$$-i\psi\left(i\frac{\partial\psi}{\partial t}+\frac{1}{2M}\nabla^2\psi\right)^*=0$$

$$-i\psi\left(-i\frac{\partial\psi^*}{\partial t}+\frac{1}{2m}\nabla^2\psi^*\right)=0$$

$$\psi\frac{\partial\psi^*}{\partial t}-i\frac{\psi}{2m}\nabla^2\psi^*=0 \qquad (1)$$

$$-i\psi^*\left(i\frac{\partial\psi}{\partial t}+\frac{1}{2m}\nabla^2\psi\right)=0.$$

$$\psi^*\frac{\partial\psi}{\partial t}-i\frac{\psi^*}{2m}\nabla^2\psi=0 \qquad (2)$$

(2) − (1)

$$\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}-\frac{i}{2m}\left(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right)=0$$

$$\underbrace{\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}} - i/2m \left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right) = 0$$

$$\frac{\partial}{\partial t} \psi^* \psi = \frac{\partial}{\partial t} |\psi|^2 = \frac{\partial \rho}{\partial t}$$

SO THIS IS:

$$\frac{\partial \rho}{\partial t} - \frac{i}{2m}\left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right) = 0$$

CONTINUITY EQUATION

PROBABILITY FLUX DENSITY

$$\bar{j} = \frac{-i}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

FOR PLANE WAVE $\psi = N e^{i\bar{p}\cdot\bar{x} - iEt}$

$$\rho = |N|^2$$

$$\bar{j} = \frac{\bar{p}}{m} |N|^2$$

4 VECTOR $(E, \bar{p})$ $(\rho, \bar{j})$

p_μ j_μ

PROBABILITY FLUX IN KLEIN-GORDON

$$E^2 = p^2 + m^2$$

MAKE OPERATOR SUBSTITUTIONS

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

$$-i\phi^* \left(\quad " \quad \right)$$

$$i\phi^* \frac{\partial^2 \phi}{\partial t^2} + i\phi^* \nabla^2 \phi = -im^2 \phi^* \phi \qquad (1)$$

$$-i\phi \left(-\frac{\partial^2 \phi^*}{\partial t^2} + \nabla^2 \phi^* = m^2 \phi^* \right)$$

$$i\phi \frac{\partial^2 \phi^*}{\partial t^2} - i\phi \nabla^2 \phi^* = -i m^2 \phi \phi^* \qquad (2)$$

$$(1) - (2)$$

$$\frac{\partial}{\partial t}\Big[\underbrace{i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)}_{\rho}\Big]$$

$$+ \nabla \cdot \Big[\underbrace{-i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right)}_{\bar{J}}\Big] = 0$$

NONRELATIVISTIC PERTURBATION

FREE PARTICLE SCHRÖDINGER

$$H_0 \phi_n = E_n \phi_n$$

$$\int_V \phi_m^* \phi_n \, d^3x = \delta_{mn}$$

FOR AN INTERACTION, MUST SOLVE

$$\left[H_0 + V(\bar{x}, t)\right]\psi = i\frac{\partial \psi}{\partial t} \quad - ①$$

POTENTIAL

ANY SOLUTION OF ① CAN BE WRITTEN AS SUM EIGENSTATES

$$\psi = \sum_n a_n(t)\, \phi_n(x)\, e^{-iE_n t} \quad ②$$

TO FIND a_n, PUT ② → ①

AND FOR H_0 TIME INDEPENDENT ONLY NEED TO KEEP TIME VARIATION

$$i\sum_n \frac{da_n}{dt}\phi_n(x) e^{-iE_n t} = \sum V(\bar{x}, t)\, a_n \phi_n(\bar{x})\, e^{-iE_n t}$$

MULTIPLY BY FINAL STATE ϕ_f^*

$\int$ OVER VOLUME

$$i \sum \frac{da_n}{dt} \int \phi_f^* \phi_n e^{-iE_n t} d^3x = \sum a_n \int \phi_f^* V \phi_n \, d^3x \, e^{-iE_n t}$$

USE $\int \phi_m^* \phi_n d^3x = \delta_{mn}$

$$\frac{da_f}{dt} = -i \sum_n a_n(t) \int \phi_f^* V \phi_n \, d^3x \cdot e^{i(E_f - E_i)t}$$

V ACTS ON EIGENSTATE i at $t = -T/2$

- $a_i\left(-\frac{T}{2}\right) = 1$
- $a_n\left(-T/2\right) = 0 \qquad n \neq i$

AND $\cdot \dfrac{da_f}{dt} = -i \int d^3x \, \phi_f^* V \phi_i \, e^{i(E_f - E_i)t}$

ASSUME V SMALL & TRANSIENT SO ALL THESE INITIAL CONDITIONS HOLD FOR ALL TIME

INTEGRATE OVER TIME

$$a_f(t) = -i \int_{-T/2}^{t} dt' \int d^3x \, \phi_f^* V \phi_i \, e^{i(E_f - E_i)t'}$$

AT TIME $t = +T/2$ AFTER INTERACTION HAS CEASED

$$T_{fi} \equiv a_f\left(\frac{T}{2}\right) = -i \int_{-T/2}^{+T/2} dt \int d^3x \left[\phi_f(\bar{x}) e^{-iE_f t}\right]^* V(\bar{x}, t) \left[\phi_i(\bar{x}) e^{-iE_i t}\right]$$

IN COVARIANT FORM, THIS IS JUST

$$T_{fi} = -i \int d^4x \, \phi_f^*(x) V(x) \phi_i(x)$$

ϕ_i^μ

ONLY GOOD FOR $a_f(t) \ll 1$

$$T_{fi} = -i \int d^4x \, \phi_f^*(x) V(x) \phi_i(x)$$

CONSIDER $V(\bar{x}, t) \equiv V(\bar{x})$ t - INDEP

$$\text{(A)} \rightarrow \quad T_{fi} = -i V_{fi} \int_{-\infty}^{+\infty} dt \, e^{i(E_f - E_i)t}$$

$$= -2\pi i V_{fi} \, \delta(E_f - E_i)$$

$$V_{fi} = \int d^3x \, \phi_f^*(\bar{x}) V(\bar{x}) \phi(x)$$

$$\delta(a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iax} dx$$

δ fn $\rightarrow$ ENERGY CONSERVED IN TRANSITION

$$\phi_i \rightarrow \phi_f$$

$$\Delta E = 0 \quad \therefore \quad \Delta t = \frac{\hbar}{\Delta E} \rightarrow \infty$$

INSTEAD OF $|T_{fi}|^2$ USE

TRANSITION PROBABILITY

PER UNIT TIME

$$W = \lim_{T \to \infty} \frac{|T_{fi}|^2}{T}$$

$$\phi = N e^{i\bar{p}\cdot\bar{x} - iEt}$$

$$\rho = i(-2iE)|N|^2 = 2E|N|^2$$

$$\bar{J} = -i(2i\bar{p})|N|^2 = 2\bar{p}|N|^2$$

IN CO-VARIANT NOTATION

KLEIN GORDON

$$(\Box^2 + m^2)\phi = (\partial^\mu \partial_\mu + m^2)\phi = 0$$

$$J^\mu = (\rho, \bar{J}) = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

$$\partial_\mu J^\mu = 0$$

COVARIANT CONTINUITY

DEFINE

$$J^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

↳ 4 VECTOR ELECTRIC CURRENT.

TAKE T_{fi} ACTING OVER TIME T

$$W = \lim_{T\to\infty} \frac{T_{fi} \cdot T_{fi}}{T}$$

$$= \lim_{T\to\infty} \left[-2\pi i V_{fi}\, \delta(E_f - E_i) \right] \left[-i V_{fi} \int_{-T/2}^{+T/2} dt\, \underbrace{e^{i(E_f - E_i)}}_{e^0} \right]$$

$$= \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{+T/2} dt$$

$$W = 2\pi |V_{fi}|^2 \delta(E_f - E_i)$$

FOR THIS TO MEAN ANYTHING HAVE TO INTEGRATE OVER SPECIFIC INITIAL + FINAL STATES

FERMI'S GOLDEN RULE

$$W = 2\pi \, |V_{fi}|^2 \, \delta(E_f - E_i)$$

INITIAL STATE → FINAL STATES

$\rho(E_f)\, dE_f$ NUMBER OF FINAL STATES $E_f \to E_f + dE_f$

DENSITY OF FINAL STATES

$$W_{fi} = 2\pi \int dE_f \, \rho(E_f) \, |V_{fi}|^2 \, \delta(E_f - E_i)$$

$$W_{fi} = 2\pi \, |V_{fi}|^2 \, \rho(E_i)$$

TRANSITION PROB

MATRIX ELEMENT

DENSITY OF FINAL STATES

A TASTE OF QUANTUM FIELD THEORY

LET'S LOOK AT A SIMPLE THEORY
(IF YOU DON'T FOLLOW THIS — IT'S OK!)

HALZEN & MARTIN CHAPTER 4
QUARKS & LEPTONS

- PRETEND THAT THERE IS A SPINLESS "ELECTRON"

FREE PARTICLE

KLEIN GORDON $$(\partial_\mu \partial^\mu + m^2)\phi = 0 \qquad (1)$$

CLASSICALLY MOTION OF $-e$ IN AN EM POTENTIAL $A^\mu = (A^0, \bar{A})$

$$p^\mu \to p^\mu + A^\mu \cdot e$$

QM $$i\partial^\mu \to i\partial^\mu + e A^\mu$$

$$(1) \Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = -V\phi$$

$$V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

$$(\partial_\mu \partial^\mu + m^2)\phi = -V\phi$$

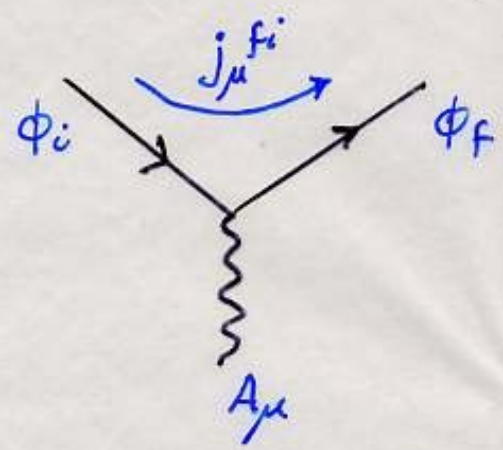
$$V = -i\underline{e}(\partial_\mu A^\mu + A^\mu \partial_\mu) - \underline{e}^2 A^2$$

IN OUR UNITS $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$

$e \gg e^2$ → KEEP TERM IN e

PERTURBATION EXPANSION

$$V = ie(\partial_\mu A^\mu + A^\mu \partial_\mu) \rightarrow \left\{ \begin{array}{l} 1^{ST} \\ \text{ORDER} \end{array} \right.$$



AMPLITUDE = $T_{fi} = -i\int \phi_f^*(x) V(x) \phi_i(x) d^4x$

$$T_{fi} = i\int \phi_f^* \, ie(\partial_\mu A^\mu + A^\mu \partial_\mu)\phi_i \, d^4x$$

$$T_{fi} = i\int \phi_f^* \, ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right)\phi_i \, d^4x$$

$$\hookrightarrow \partial_\mu \left(A^\mu \phi_i\right)$$

INTEGRATING BY PARTS

$$\int \phi_f^* \partial_\mu (A^\mu \phi_i)\, d^4x = -\int \partial_\mu (\phi_f^*) A^\mu \phi_i \, d^4$$

$$T_{fi} = -i\int j_\mu^{fi} A^\mu d^4x$$

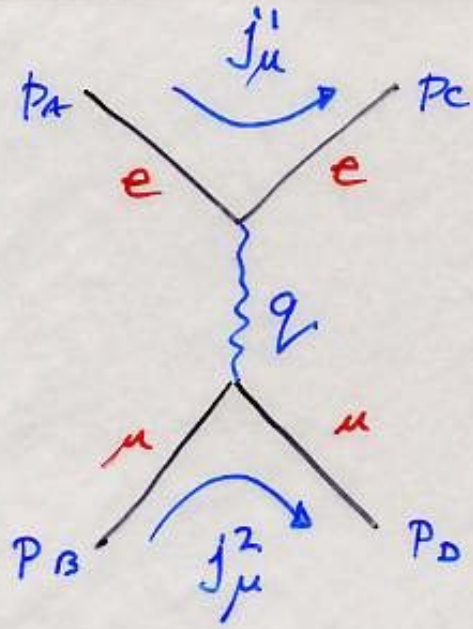
$$j_\mu^{fi}(x) \equiv -ie\left(\phi_f^*(\partial_\mu \phi_i) - (\partial_\mu \phi_f^*)\phi_i\right)$$

EM CURRENT

$$\phi_i(x) = N_i e^{-ip_i \cdot x} \qquad (i, f)$$

$$j_\mu^{fi} = -e N_i N_f (p_i + p_f)_\mu e^{i(p_f - p_i)\cdot x}$$

PHASE CHANGE AT VERTEX



$ee \to \mu\mu$

MUON IS SOURCE OF A_μ

MAXWELL $\quad \Box^2 A^\mu = j_2^\mu \quad -(1)$

BUT $j_2^\mu = -e N_B N_D (p_D + p_B)^\mu e^{i(p_D - p_B)\cdot x}$

AND $\Box^2 e^{iq\cdot x} = -q^2 e^{iq\cdot x}$ PHOTON

SOLUTION OF (1) is

$$A^\mu = -\frac{1}{q^2} j_2^\mu \quad ; \quad q = p_D - p_B$$

$$T_{fi} = -i \int j_\mu^1(x) \left(\frac{-1}{q^2}\right) j_2^\mu(x)\, d^4$$

$\Rightarrow$ SUBSTITUTE EXPLICIT CURRENTS

$$T_{fi} = -i\, N_A N_B N_C N_D (2\pi)^4 \underbrace{\delta^4(p_D + p_C - p_B - p_A)}_{\text{4-MOMENTUM CONSERV}} \mathcal{M}$$

$\rightarrow \int$ OVER μ INDICES

$$-i\mathcal{M} = i e (p_A + p_C)^\mu$$

$$\times \frac{-i g_{\mu\nu}}{q^2}$$

$$\times i e (p_B + p_D)^\nu$$