ANGULAR MOMENTUM & ISOSPIN

- USED TO THE IDEA THAT IN ADDITION TO REAL SPACE WE CAN INTRODUCE IDEA OF ABSTRACT SPACE

- LET'S LOOK AT INVARIANCE UNDER ROTATIONS IN REAL SPACE

  → CONSERVATION OF ANGULAR MOMENTUM (SPIN)

- THEN GENERALIZE THIS IDEA TO ABSTRACT "ISOSPACE"

  → CONSERVATION OF ISOSPIN

ANGULAR MOMENTUM

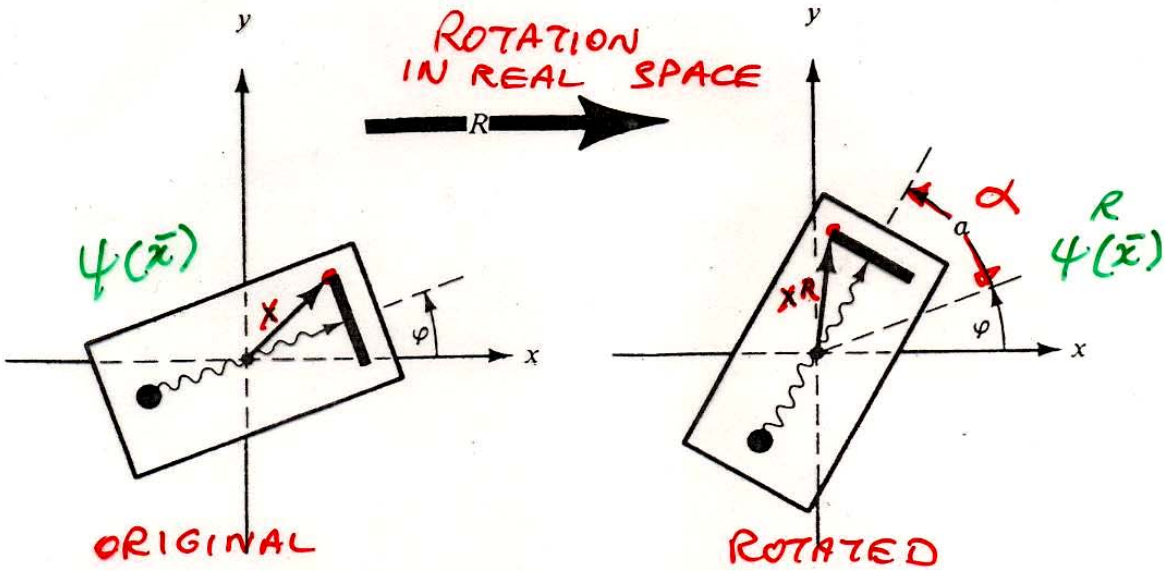

Fig. 8.1. Rotation around the z axis. The angle φ fixes the position of the original equipment axis; it does not denote a rotation. The equipment is rotated about the z axis by an angle α. Invariance under rotation means that the outcome of the experiment is not affected by the rotation.

- WORK IN $x-y$ FOR SIMPLICITY
- ORIGINAL WAVE FN $\psi(\bar{x})$
- ROTATION

POINT $\bar{x} \xrightarrow{R} \bar{x}^R$

$$\bar{x}^R = R_z(\alpha)\,\bar{x}$$

ROTATION

$$\psi^R(\bar{x}) = \mathcal{U}_z(\alpha)\,\psi(\bar{x})$$

TRANSFORMED WAVE FN AT $\bar{x}$

ANGULAR MOMENTUM

$$\bar{x}^R = R_z(\alpha)\,\bar{x}$$

$$\psi^R(\bar{x}) = U_z(\alpha)\,\psi(\bar{x}) \qquad \text{A}$$

USE INVARIANCE TO DETERMINE FORM of $U_z(\alpha)$

- IF STATE OF SYSTEM UNAFFECTED BY ROTATION

$$\psi(\bar{x}) = \psi^R(\bar{x}^R) \qquad \text{B}$$

A $\rightarrow$ SAME POINT $\bar{x}$ $\quad \psi^R(\bar{x}) \Leftrightarrow \psi(\bar{x})$

B $\rightarrow$ ROTATED $\bar{x}$ $\quad \psi(\bar{x}) \Leftrightarrow \psi^R(\bar{x}^R)$

- CAN FIND $U_z(\alpha)$ IF $\psi^R(\bar{x}^R) \Leftrightarrow \psi^R(\bar{x})$

- BUILD ROTATION FROM SERIES OF INFINITESIMAL ROTATIONS

$$\psi^R(\bar{x}^R) = \psi^R(\bar{x}) + \frac{\partial \psi^R(\bar{x})}{\partial \phi} \cdot \partial\alpha$$

PROPERTY OF INFINITESIMAL

$\Longleftrightarrow$ PROPERTY OF FINITE

$$\psi^R(\bar{x}^R) = \psi^R(\bar{x}) + \frac{\partial \psi^R(\bar{x})}{\partial \phi}\,\delta\alpha$$

INFINITESIMAL ROTATION

$$= \left(1 + \delta\alpha \frac{\partial}{\partial \phi}\right)\psi^R(\bar{x})$$

INVERT

$$\left(1 - \delta\alpha \frac{\partial}{\partial \phi}\right)\psi^R(\bar{x}^R) = \left(1 - \delta\alpha \frac{\partial}{\partial \phi}\right)\left(1 + \delta\alpha \frac{\partial}{\partial \phi}\right) \cdot \psi^R(\bar{x})$$

ROTATIONAL INVARIANCE $\Rightarrow$ $\Downarrow$

$$\psi(\bar{x})$$

MOST IMPORTANT PHYSICAL STEP

$$\left(1 - \delta\alpha \frac{\partial}{\partial \phi}\right)\psi(\bar{x}) = \left(1 - \delta\alpha \frac{\partial}{\partial \phi} + \delta\alpha \frac{\partial}{\partial \phi} - (\delta\alpha)^2 \frac{\partial}{\partial \phi}\cdot\frac{\partial}{\partial \phi}\right)\psi^R(x)$$

$$(\delta\alpha)^2 \to 0$$

$$\psi^R(\bar{x}) = \left(1 - \delta\alpha \frac{\partial}{\partial \phi}\right)\psi(\bar{x})$$

cf $\psi^R(\bar{x}) = \mathcal{U}_z(\alpha)\,\psi(\bar{x})$

$$\psi^R(\bar{x}) = \left(1 - \delta\alpha \frac{\partial}{\partial\phi}\right)\psi(\bar{x})$$

$$\psi^R(\bar{x}) = \mathcal{U}_z(\alpha)\,\psi(\bar{x})$$

$$\mathcal{U}_z(\alpha) = \left(1 - \delta\alpha \frac{\partial}{\partial\phi}\right)$$

- SHOWED EARLIER THAT FOR UNITARY TRANSFORMATION $\mathcal{U} = e^{i\varepsilon F}$
- INFINITESIMAL UNITARY TRANSFORM IS:

$$\mathcal{U} = 1 + i\,\varepsilon\,F$$

$$\varepsilon \Leftrightarrow \delta\alpha$$

$$F = i\frac{\partial}{\partial\phi}$$

FROM Q. M. $L_z = -i\hbar\frac{\partial}{\partial\phi}$ Z-COMP ANGULAR MOMENTUM

$$F = i\frac{\partial}{\partial\phi} = \frac{-L_z}{\hbar}$$

$$\mathcal{U} = 1 + i\,\delta\alpha\, F$$

$$F = i\frac{\partial}{\partial\phi} = -\frac{L_z}{\hbar} \Rightarrow -J_3$$

INFINITESIMAL ROTATION $\Rightarrow$ FINITE ROTATION

$$\mathcal{U}(\theta) = \left[\mathcal{U}(\delta\alpha)\right]^n$$

$$= \left(1 - i\frac{\theta}{n}J_3\right)^n \qquad \frac{\theta}{n} \to \text{INFINITES ROT.}$$

$$\xrightarrow[n\to\infty]{} e^{-i\theta J_3}$$

$$\left[J_1, J_2\right] = i\,J_3$$

$$\left[J_i, J_j\right] = i\,\varepsilon_{ijk}\,J_k$$

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

$$[J^2, J_i] = 0 \qquad i = 1,2,3$$

SQUARE OF TOTAL SPIN
(CASIMIR OPERATOR)

$$J^2 |j\, m\rangle = j(j+1)\, |j\, m\rangle$$

$$J_3 |j\, m\rangle = m\, |j, m\rangle$$

→ CAN HAVE THIS IN ANY ABSTRACT SPACE

$$U(\theta) = \exp\left(-\frac{\bar{\sigma} \cdot \bar{J}}{2} \right)$$

↑ PROJECTION

GROUP SU(2)

$$U(\theta) = e^{-i\theta J_3}$$

IS THE GROUP OF ROTATIONS AROUND AN AXIS IN 3-d SPACE

θ — PARAMETER

J_3 — GENERATOR

GROUP OF ROTATIONS ABOUT ANY AXIS IN 3-d

$$J_3 \rightarrow J_i$$

CAN BE "REPRESENTED" BY PAULI SPIN MATRICES

$$J_i = \frac{1}{2}\sigma_i \qquad i = 1, 2, 3$$

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$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

USE EIGENVECTORS OF J_3 AS BASIS

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$U(\theta_i) = e^{-i\theta_i \sigma_i/2}$$

- ALL 2×2 MATRICES $\mathcal{U}(2)$
- σ_i ARE GENERATORS

  HERMITIAN, TRACE $= 0$

$$\therefore \det\left(e^{i\sigma}\right) = e^{i\,\mathrm{Tr}(\sigma)} = 1$$

$\sigma_i \rightarrow$ SUBGROUP $\rightarrow$ $SU(2)$

RESUMÉ

SHOWED $[H, U] = 0$ INVARIANCE UNDER ROTATION

$$\Downarrow$$

$[H, F] = 0$ ROTATION GENERATOR CONSERVED

$$\Downarrow$$

$[L_z, H] = 0$ ANGULAR MOMENTUM CONSERVED

$[J_3, H] = 0$ GENERALIZE TO ANY ABSTRACT SPACE

ISOSPIN SYMMETRY

- THE STRONG INTERACTION IS EXPERIMENTALLY INDEPENDENT OF ELECTRIC CHARGE

- FOR THE STRONG INTERACTION, IT LOOKS LIKE

$$p, n$$
$$\pi^+ \quad \pi^- \quad \pi^0$$

BEHAVE AS IF THEY ARE SAME PARTICLE, AFTER ALLOWING FOR ELECTRIC CHARGE

$$\sigma(pn \to pn) = \sigma(nn \to nn)$$
$$= \sigma(pp \to pp)$$

$$\sigma(\pi^+ p \to \pi^+ p) = \sigma(\pi^- n \to \pi^- n)$$

PARTICLES SEEM TO COME IN

ISOMULTIPLETS

OF VERY SIMILAR MASSES

				I_3
+	p	938.27	MeV	$+\frac{1}{2}$
0	n	939.57	MeV	$-\frac{1}{2}$
+	π^+	139.57	MeV	+1
0	π^0	134.96	MeV	0
−	π^-	139.57	MeV	−1

- HYPOTHESIS
  - NUCLEON CAN HAVE 2 ORIENTATIONS IS ISOSPACE

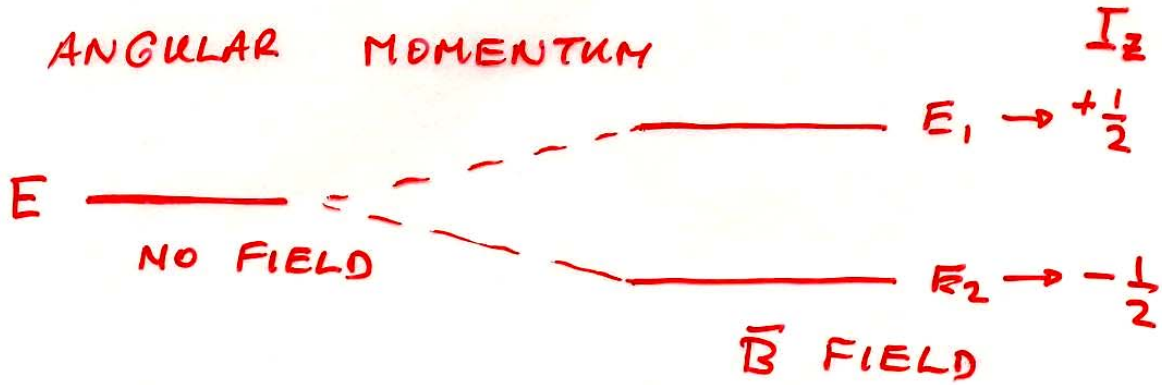
    + → ISOSPIN UP

    − → ISOSPIN DOWN

  - PION 3 ORIENTATIONS

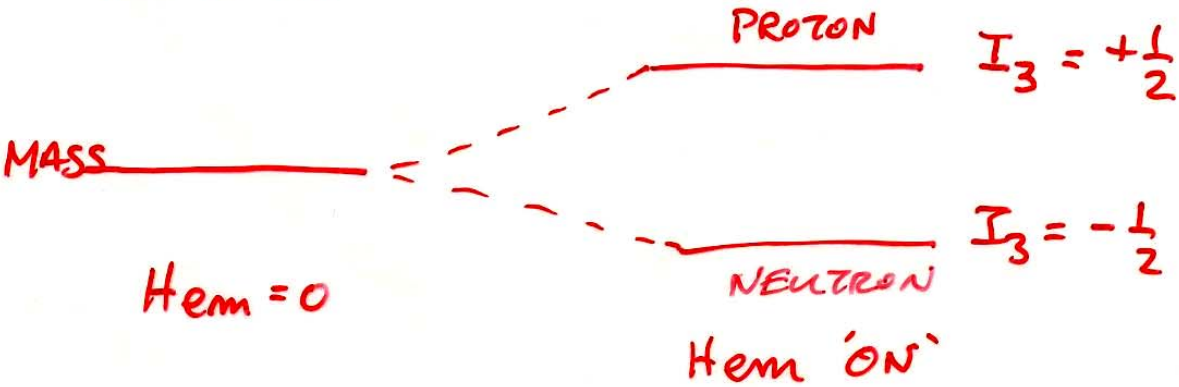
IT IS LIKE ANGULAR MOMENTUM DEGENERACY FOR SPIN

ANGULAR MOMENTUM



- PRESENCE OF ELECTROMAGNETISM "BREAKS DEGENERACY IN ISO SPACE"

ISO SPIN



- PROTON AND NEUTRON ARE DIFFERENT ISOSPIN STATES OF NEUCLON

$$|\text{NUCLEON}\rangle = |I, I_3\rangle \rightarrow \begin{array}{ll} |\tfrac{1}{2}\ +\tfrac{1}{2}\rangle & p \\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle & n \end{array}$$

- SAME FOR PION

$$|\text{PION}\rangle = |I, I_3\rangle \rightarrow \begin{array}{ll} |1, +1\rangle & \pi^+ \\ |1, 0\rangle & \pi^0 \\ |1, -1\rangle & \pi^- \end{array}$$

IF ASSUME ISOSPIN CONSERVED IN STRONG INTERACTIONS, RELATE MANY PROCESS

- WILL NOT PURSUE FURTHER STRONG ISOSPIN IS JUST A REFLECTION OF

- $m_u \approx m_d$

- QUARK INTERACTIONS DEPEND ON COLOUR NOT ELECTRIC CHARGE

→ BUT IT SUGGESTS IDEA OF A SYMMETRY BASED ON ROTATIONS IN AN ABSTRACT SPACE

$$\begin{pmatrix} e^- \\ \nu_e \end{pmatrix} \quad \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix} \quad \begin{pmatrix} \tau^- \\ \nu_\mu \end{pmatrix}$$

→ COULD THIS BE SOMETHING TO DO WITH "WEAK ISOSPIN DOUBLETS"