THE SUBATOMIC ZOO — PARTICLE CLASSIFICATION

- MANY PARTICLES
  - MOST ARE COMPOSITE
  - ONLY QUARKS, LEPTONS, GAUGE BOSONS ARE ELEMENTARY — NO KNOWN INTERNAL STRUCTURE
- MUST UNDERSTAND WHAT ATTRIBUTES LABEL PARTICLES
- ATTRIBUTES WILL BE • INVARIANT QUANTITIES
  OR • CONSERVED QUANTITIES
- REST MASS IS AN INVARIANT (NOT CONSERVED)
- ONE CAN ASK SENSIBLE QUESTIONS ABOUT INVARIANT QUANTITIES

  WHY is $m_\nu \ll m_e \ll m_p \ll m_Z$ ..... ?

  WHY ARE ALL MASSES NOT $= 0$?

LIGHT $I = 1$ MESONS

π^0	111
π^+	211
$a_0(980)^0$	9000111
$a_0(980)^+$	9000211
$\pi(1300)^0$	100111
$\pi(1300)^+$	100211
$a_0(1450)^0$	10111
$a_0(1450)^+$	10211
$\pi(1800)^0$	9010111
$\pi(1800)^+$	9010211
$\rho(770)^0$	113
$\rho(770)^+$	213
$b_1(1235)^0$	10113
$b_1(1235)^+$	10213
$a_1(1260)^0$	20113
$a_1(1260)^+$	20213
$\pi_1(1400)^0$	9000113
$\pi_1(1400)^+$	9000213
$\rho(1450)^0$	100113
$\rho(1450)^+$	100213
$\pi_1(1600)^0$	9010113
$\pi_1(1600)^+$	9010213
$a_1(1640)^0$	9020113
$a_1(1640)^+$	9020213
$\rho(1700)^0$	30113
$\rho(1700)^+$	30213
$\rho(1900)^0$	9030113
$\rho(1900)^+$	9030213
$\rho(2150)^0$	9040113
$\rho(2150)^+$	9040213
$a_2(1320)^0$	115
$a_2(1320)^+$	215
$\pi_2(1670)^0$	10115
$\pi_2(1670)^+$	10215
$a_2(1700)^0$	9000115
$a_2(1700)^+$	9000215
$\pi_2(2100)^0$	9010115
$\pi_2(2100)^+$	9010215
$\rho_3(1690)^0$	117
$\rho_3(1690)^+$	217
$\rho_3(1990)^0$	9000117
$\rho_3(1990)^+$	9000217
$\rho_3(2250)^0$	9010117
$\rho_3(2250)^+$	9010217
$a_4(2040)^0$	119
$a_4(2040)^+$	219

LIGHT $I = 0$ MESONS
($u\bar{u}$, $d\bar{d}$, and $s\bar{s}$ Admixtures)

η	221
$\eta'(958)$	331
$f_0(600)$	9000221
$f_0(980)$	9010221
$\eta(1295)$	100221
$f_0(1370)$	10221
$\eta(1405)$	9020221
$\eta(1475)$	100331
$f_0(1500)$	9030221
$f_0(1710)$	10331
$\eta(1760)$	9040221
$f_0(2020)$	9050221
$f_0(2100)$	9060221
$f_0(2200)$	9070221
$\eta(2225)$	9080221
$\omega(782)$	223
$\phi(1020)$	333
$h_1(1170)$	10223
$f_1(1285)$	20223
$h_1(1380)$	10333
$f_1(1420)$	20333
$\omega(1420)$	100223
$f_1(1510)$	9000223
$h_1(1595)$	9010223
$\omega(1650)$	30223
$\phi(1680)$	100333
$f_2(1270)$	225
$f_2(1430)$	9000225
$f_2'(1525)$	335
$f_2(1565)$	9010225
$f_2(1640)$	9020225
$\eta_2(1645)$	10225
$f_2(1810)$	9030225
$\eta_2(1870)$	10335
$f_2(1910)$	9040225
$f_2(1950)$	9050225
$f_2(2010)$	9060225
$f_2(2150)$	9070225
$f_2(2300)$	9080225
$f_2(2340)$	9090225
$\omega_3(1670)$	227
$\phi_3(1850)$	337
$f_4(2050)$	229
$f_J(2220)$	9000229
$f_4(2300)$	9010229

STRANGE MESONS

K_L^0	130
K_S^0	310
K^0	311
K^+	321
$K_0^*(800)^0$	9000311
$K_0^*(800)^+$	9000321
$K_0^*(1430)^0$	10311
$K_0^*(1430)^+$	10321
$K(1460)^0$	100311
$K(1460)^+$	100321
$K(1830)^0$	9010311
$K(1830)^+$	9010321
$K_0^*(1950)^0$	9020311
$K_0^*(1950)^+$	9020321
$K^*(892)^0$	313
$K^*(892)^+$	323
$K_1(1270)^0$	10313
$K_1(1270)^+$	10323
$K_1(1400)^0$	20313
$K_1(1400)^+$	20323
$K^*(1410)^0$	100313
$K^*(1410)^+$	100323
$K_1(1650)^0$	9000313
$K_1(1650)^+$	9000323
$K^*(1680)^0$	30313
$K^*(1680)^+$	30323
$K_2^*(1430)^0$	315
$K_2^*(1430)^+$	325
$K_2(1580)^0$	9000315
$K_2(1580)^+$	9000325
$K_2(1770)^0$	10315
$K_2(1770)^+$	10325
$K_2(1820)^0$	20315
$K_2(1820)^+$	20325
$K_2^*(1980)^0$	9010315
$K_2^*(1980)^+$	9010325
$K_2(2250)^0$	9020315
$K_2(2250)^+$	9020325
$K_3^*(1780)^0$	317
$K_3^*(1780)^+$	327
$K_3(2320)^0$	9010317
$K_3(2320)^+$	9010327
$K_4^*(2045)^0$	319
$K_4^*(2045)^+$	329
$K_4(2500)^0$	9000319
$K_4(2500)^+$	9000329

CHARMED MESONS

D^+	411
D^0	421
$D_0^*(2400)^+$	10411
$D_0^*(2400)^0$	10421
$D^*(2010)^+$	413
$D^*(2007)^0$	423
$D_1(2420)^+$	10413
$D_1(2420)^0$	10423
$D_1(H)^+$	20413
$D_1(2430)^0$	20423
$D_2^*(2460)^+$	415
$D_2^*(2460)^0$	425
D_s^+	431
$D_{s0}^*(2317)^+$	10431
D_s^{*+}	433
$D_{s1}(2536)^+$	10433
$D_{s1}(2460)^+$	20433
$D_{s2}^*(2573)^+$	435

BOTTOM MESONS

B^0	511
B^+	521
B_0^{*0}	10511
B_0^{*+}	10521
B^{*0}	513
B^{*+}	523
$B_1(L)^0$	10513
$B_1(L)^+$	10523
$B_1(H)^0$	20513
$B_1(H)^+$	20523
B_2^{*0}	515
B_2^{*+}	525
B_s^0	531
B_{s0}^{*0}	10531
B_s^{*0}	533
$B_{s1}(L)^0$	10533
$B_{s1}(H)^0$	20533
B_{s2}^{*0}	535
B_c^+	541
B_{c0}^{*+}	10541
B_c^{*+}	543
$B_{c1}(L)^+$	10543
$B_{c1}(H)^+$	20543
B_{c2}^{*+}	545

$c\bar{c}$ MESONS

$\eta_c(1S)$	441
$\chi_{c0}(1P)$	10441
$\eta_c(2S)$	100441
$J/\psi(1S)$	443
$h_c(1P)$	10443
$\chi_{c1}(1P)$	20443
$\psi(2S)$	100443
$\psi(3770)$	30443
$\psi(4040)$	9000443
$\psi(4160)$	9010443
$\psi(4415)$	9020443
$\chi_{c2}(1P)$	445
$\chi_{c2}(2P)$	100445

$b\bar{b}$ MESONS

$\eta_b(1S)$	551
$\chi_{b0}(1P)$	10551
$\eta_b(2S)$	100551
$\chi_{b0}(2P)$	110551
$\eta_b(3S)$	200551
$\chi_{b0}(3P)$	210551
$\Upsilon(1S)$	553
$h_b(1P)$	10553
$\chi_{b1}(1P)$	20553
$\Upsilon_1(1D)$	30553
$\Upsilon(2S)$	100553
$h_b(2P)$	110553
$\chi_{b1}(2P)$	120553
$\Upsilon_1(2D)$	130553
$\Upsilon(3S)$	200553
$h_b(3P)$	210553
$\chi_{b1}(3P)$	220553
$\Upsilon(4S)$	300553
$\Upsilon(10860)$	9000553
$\Upsilon(11020)$	9010553
$\chi_{b2}(1P)$	555
$\eta_{b2}(1D)$	10555
$\Upsilon_2(1D)$	20555
$\chi_{b2}(2P)$	100555
$\eta_{b2}(2D)$	110555
$\Upsilon_2(2D)$	120555
$\chi_{b2}(3P)$	200555
$\Upsilon_3(1D)$	557
$\Upsilon_3(2D)$	100557

LIGHT BARYONS

p	2212
n	2112
Δ^{++}	2224
Δ^+	2214
Δ^0	2114
Δ^-	1114

STRANGE BARYONS

Λ	3122
Σ^+	3222
Σ^0	3212
Σ^-	3112
Σ^{*+}	3224[d]
Σ^{*0}	3214[d]
Σ^{*-}	3114[d]
Ξ^0	3322
Ξ^-	3312
Ξ^{*0}	3324[d]
Ξ^{*-}	3314[d]
Ω^-	3334

CHARMED BARYONS

Λ_c^+	4122
Σ_c^{++}	4222
Σ_c^+	4212
Σ_c^0	4112
Σ_c^{*++}	4224
Σ_c^{*+}	4214
Σ_c^{*0}	4114
Ξ_c^+	4232
Ξ_c^0	4132
$\Xi_c'^+$	4322
$\Xi_c'^0$	4312
Ξ_c^{*+}	4324
Ξ_c^{*0}	4314
Ω_c^0	4332
Ω_c^{*0}	4334
Ξ_{cc}^+	4412
Ξ_{cc}^{++}	4422
Ξ_{cc}^{*+}	4414
Ξ_{cc}^{*++}	4424
Ω_{cc}^+	4432
Ω_{cc}^{*+}	4434
Ω_{ccc}^{++}	4444

PENTAQUARKS

Θ^+	9221132
Φ^{--}	9331122

BOTTOM BARYONS

Λ_b^0	5122
Σ_b^-	5112
Σ_b^0	5212
Σ_b^+	5222
Σ_b^{*-}	5114
Σ_b^{*0}	5214
Σ_b^{*+}	5224
Ξ_b^-	5132
Ξ_b^0	5232
$\Xi_b'^-$	5312
$\Xi_b'^0$	5322
Ξ_b^{*-}	5314
Ξ_b^{*0}	5324
Ω_b^-	5332
Ω_b^{*-}	5334
Ξ_{bc}^0	5142
Ξ_{bc}^+	5242
$\Xi_{bc}'^0$	5412
$\Xi_{bc}'^+$	5422
Ξ_{bc}^{*0}	5414
Ξ_{bc}^{*+}	5424
Ω_{bc}^0	5342
$\Omega_{bc}'^0$	5432
Ω_{bc}^{*0}	5434
Ω_{bcc}^+	5442
Ω_{bcc}^{*+}	5444
Ξ_{bb}^-	5512
Ξ_{bb}^0	5522
Ξ_{bb}^{*-}	5514
Ξ_{bb}^{*0}	5524
Ω_{bb}^-	5532
Ω_{bb}^{*-}	5534
Ω_{bbc}^0	5542
Ω_{bbc}^{*0}	5544
Ω_{bbb}^-	5554

IF I COULD REMEMBER THE NAMES OF ALL THESE PARTICLES, THEN COULD BE A ZOOLOGIST — E. FERMI

CONSERVED QUANTITY

- A GOOD EXAMPLE IS ELECTRIC CHARGE
- IN A PROCESS LIKE (OR ANY PROCESS!)

$$\pi^- p \rightarrow \pi^0 n \pi^+ \pi^-$$

TOTAL ELECTRIC CHARGE IS SAME BEFORE AND AFTER COLLISION — WHY? # PARTICLES CHANGES

- THE ELECTRIC CHARGE ON A PROTON IS EXACTLY EQUAL TO THAT ON AN ELECTRON WHY?

- WE BELIEVE THAT CONSERVED QUANTITIES REFLECT SYMMETRIES

CONSERVATION OF ELECTRIC CHARGE

$\Rightarrow$ LOCAL GAUGE INVARIANCE

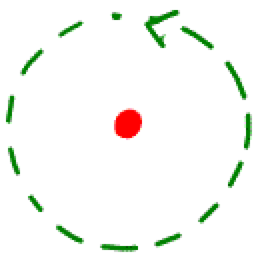
$\rightarrow$ I'LL EXPLAIN THAT!

ANGULAR MOMENTUM

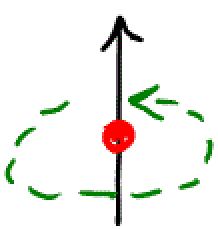
- GOOD EXAMPLE OF CONSERVED QUANTITY
- CLASSICALLY ARISES FROM A SYMMETRY

— ISOLATED SYSTEMS INVARIANT UNDER ROTATIONS

- TWO KINDS OF ANGULAR MOMENTUM

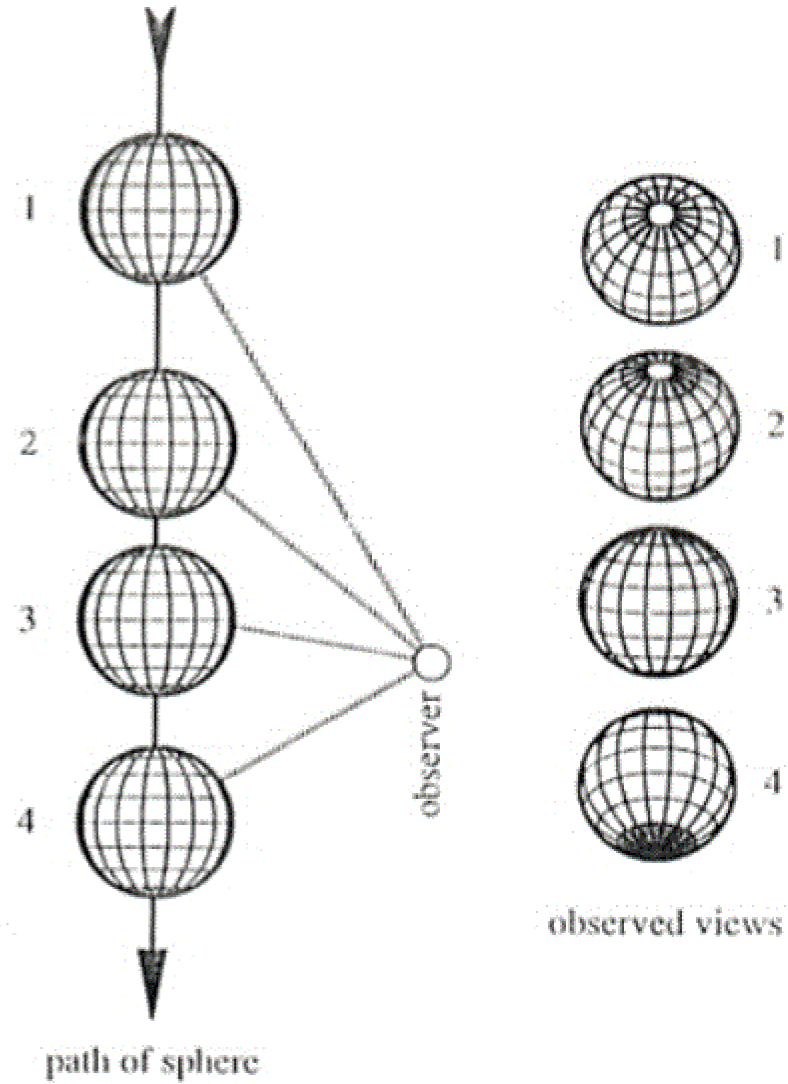


- ORBITAL (RELATIVE) ANGULAR MOMENTUM
- HAS A CLASSICAL ANALOG.



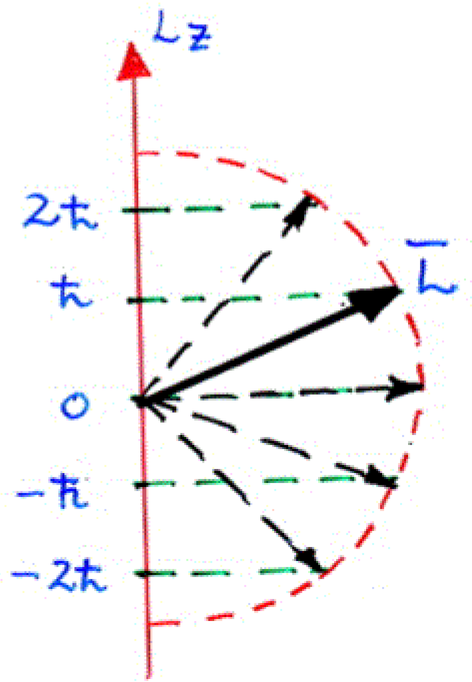
- INTRINSIC ANGULAR MOMENTUM
- THIS APPEARS TO HAVE A CLASSICAL ANALOG — A CLASSICAL PARTICLE CAN HAVE ARBITRARY SPIN

- IN SUBATOMIC PHYSICS SPIN IS AN ATTRIBUTE OF A PARTICLE — ALL ELECTRONS HAVE SPIN $\hbar/2$



- IN SPECIAL RELATIVITY THERE IS NO PREFERED FRAME
- AN ARBITRARY OBSERVER CAN NOT DISTINGUISH ORBITAL FROM INTRINSIC ANGULAR MOMENTUM
- INTRINSIC ANGULAR MOMENTUM IS IMPLIED BY SPECIAL RELATIVITY.

QUANTIZATION OF ORBITAL ANGULAR MOMENTUM



- QUANTUM MECHANICALLY $\bar{L}$ CAN ONLY ASSUME CERTAIN SPATIAL ORIENTATIONS L_z QUANTIZED

$|\bar{L}|$ QUANTIZED

$$\bar{L}^2 |\psi_{\ell m}\rangle = \ell(\ell+1)\hbar^2 |\psi_{\ell m}\rangle$$

ANGULAR MOMENTUM EIGENSTATE

VALUE OF TOTAL ANGULAR MOMENTUM

$\ell \rightarrow$ MAGNITUDE

$m \rightarrow$ Z COMPONENT

$$L_z |\psi_{\ell m}\rangle = m\hbar |\psi_{\ell m}\rangle$$

$$L^2 |\psi_{\ell m}\rangle = \ell(\ell+1)\hbar^2 |\psi_{\ell m}\rangle \qquad \ell = 0, 1, 2, \cdots\cdots$$

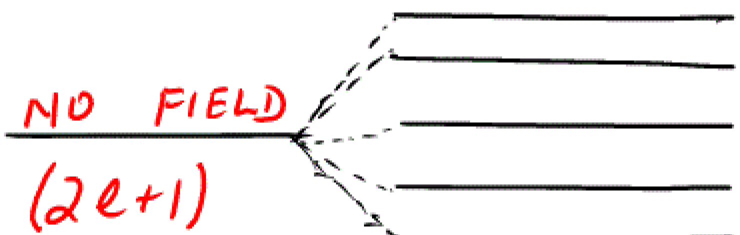
$$L_z |\psi_{\ell m}\rangle = m\hbar |\psi_{\ell m}\rangle \qquad m_\ell = -\ell, -\ell+1 \cdots -1, 0, +1 \cdots\cdots \ell-1, \ell$$

• FOR A GIVEN ℓ THERE ARE $(2\ell+1)$ ← ODD ORIENTATIONS IN A MAGNETIC FIELD → ℓ INTEGER

ATOMIC SPECTRAL LINES IN A MAGNETIC FIELD

THIS WAS A SURPRISE !

NO FIELD



$(2\ell+1)$ DEGENERATE STATES

FIELD ON
ODD NUMBER OF SPLIT LINES

FIELD OFF

FIELD ON

2 ORIENTATIONS

$(2\ell+1) = 2$ ← EVEN

$\ell = 1/2$ → 1/2 UNIT OF $\hbar$

- CLASSICAL INTERPRETATION — ELECTRON SPINNING ON ITS AXIS
  - HOW COULD A POINT PARTICLE SPIN?
  - PROTON HAS SPIN $\hbar/2$. CALCULATE THE VELOCITY OF ITS PERIPHERY.
- BUT SPIN $\frac{1}{2}$ $\underbrace{\text{BEHAVES LIKE ANGULAR MOMENTUM}}$

WHAT DOES THIS MEAN?

QUANTUM OPERATORS OBEY COMMUTATION RELATION

$$[L_x, L_y] = L_x L_y - L_y L_x = i\hbar L_z$$

$$[L_y, L_z] = [L_z, L_x] = i\hbar L_{x,y}$$

$$\left[\bar{L}^2, \begin{matrix} L_x \\ L_y \\ L_z \end{matrix} \right]$$

← ANY SET OF OPERATORS OBEYING THIS "ALGEBRA" "BEHAVES LIKE ANGULAR MOMENTUM"

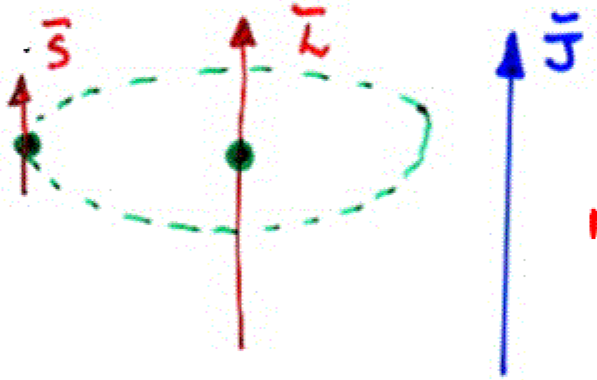
← "GROUP ALGEBRA" REFLECTION OF SYMMETRY

↓

ROTATIONAL INVARIANCE

ORBITAL ANGULAR MOMENTUM, SPIN

ADDITION OF ANGULAR MOMENTUM



- ANY SYSTEM INVOLVING SEVERAL ANGULAR MOMENTUM — TOTAL ANGULAR MOMENTUM

$\bar{J} = \bar{L} + \bar{S}$

- SIMPLE EXAMPLE IS A HYDROGEN ATOM

- SAY HAVE TWO STATES ADDING $|J_1>$, $|J_2>$

$$\bar{J}_1^2 |j_1, m_1> = j_1(j_1+1)\hbar^2 |j_1, m_1>$$

$$J_{z_1} |j_1, m_1> = m_1 \hbar |j_1, m_1>$$

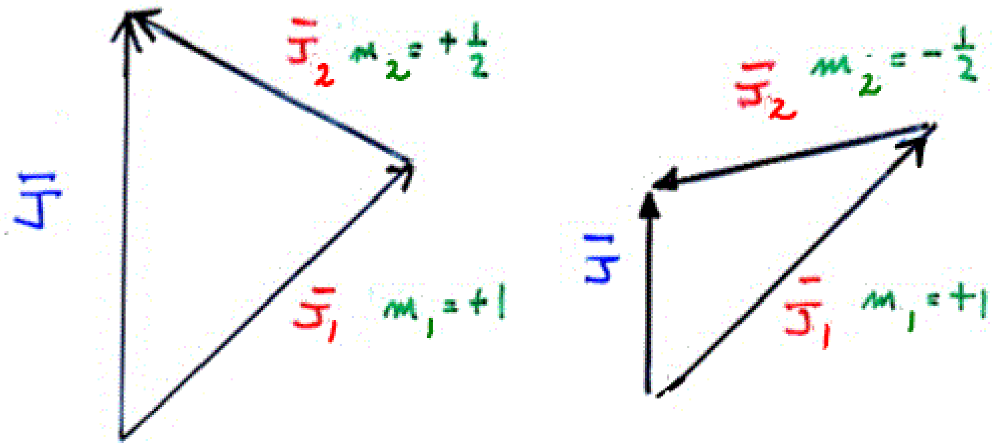
$\hbar \to 1$ LAZY

- OBVIOUSLY TOTAL STATE $\bar{J} = \bar{J}_1 + \bar{J}_2$ WILL HAVE Z-COMPONENT

$$J_z |?(m_1+m_2)> = (m_1+m_2)|?(m_1+m_2)>$$

$m = m_1 + m_2 \leftarrow$ Z COMPONENTS ADD

VECTOR ADDITION OF ANGULAR MOMENTA



$m = m_1 + m_2 = (1 + \frac{1}{2}) = 3/2$

$m = m_1 + m_2 = (1 - 1/2) = 1/2$

$$\bar{J} = \bar{J}_1 + \bar{J}_2$$

- Z COMPONENTS ADD
- MAGNITUDES DO NOT

$j_1 = 1 + j_2 = 1/2$

CAN ADD TO

$j = 3/2$ OR $j = 1/2$

$$|j_1, m_1\rangle |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j\,j_1 j_2}_{m\,m_1 m_2} |j, m\rangle$$

CONTRIBUTING STATES

$\sqrt{\ }$ OF PROBABILITY OF $|j, m\rangle$ OCCURING

CLEBSCH-GORDAN COEFFICIENTS

TOTAL STATE

• SIMPLE EXAMPLE $\bar{J} = \bar{L} + \bar{S}$ $\ell = 1,\ s = \frac{1}{2}$

• THIS MEANS ADD $\left|1\ \begin{matrix}+1\\0\\-1\end{matrix}\right\rangle$ TO $\left|\frac{1}{2}\ \begin{matrix}+\frac{1}{2}\\-\frac{1}{2}\end{matrix}\right\rangle$

$|j_1\, m_1\rangle$ $\qquad |j_2\, m_2\rangle$

$|\ell,\ \ell_z\rangle$ $\qquad |s,\ s_z\rangle$

• FROM ADDING Z-COMPONENTS, SEE THAT $\ell = 1$ ADDING TO $s = 1/2$ COULD GIVE

$$\left|\tfrac{1}{2}\ \pm 1/2\right\rangle$$

OR

$$\left|\tfrac{3}{2}\ \begin{matrix}\pm 3/2\\ \pm 1/2\end{matrix}\right\rangle$$

WHAT ARE THE PROBABILITIES OF EACH OF THESE POSSIBLE OUTCOMES ?

- WE ARE COMBINING $\left|1 \;\begin{matrix}+1\\0\\-1\end{matrix}\right\rangle + \left|\tfrac{1}{2} \;\begin{matrix}+1/2\\-1/2\end{matrix}\right\rangle$

- IF THEY ARE PARALLEL $\rightarrow$ ONLY ONE POSSIBILITY

$\uparrow\uparrow$

$$|1 \; +1\rangle|1/2 \; +1/2\rangle = 1 \left|\tfrac{3}{2} \; +\tfrac{3}{2}\right\rangle$$

- IF THEY ARE ANTI-PARALLEL $\uparrow\downarrow$

$$|1 \; +1\rangle\left|\tfrac{1}{2} \; -1/2\right\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2} \; +\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2} \; +1/2\right\rangle$$

$\sqrt{\text{PROBABILITY}}$

THIS EXAMPLE IS AN $m = +\frac{1}{2}$ STATE WHICH CAN B $J = 3/2$ OR $j = 1/2$ WITH $\sqrt{}$ GIVING PROBABILITY

POSSIBLE TOTALS

PROBABILITIES

	j	j
	m	m
$m_1\ m_2$	CLEBSCH GORDAN COEFF	
$m_1\ m_2$		

CONTRIBUTING STATES

$|1\ {}^{+1}_{\ 0}\rangle|\tfrac{1}{2}\ \pm\tfrac{1}{2}\rangle$

$\downarrow$

$\underline{1} \times 1/2$

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

FROM CLEBSH – GORDAN TABLE

$$|1,0\rangle\left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\ +\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle$$

$$|1,0\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$|1,-1\rangle\left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle = \sqrt{1/3}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$|1,-1\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = 1\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$$

CAN ALSO USE TABLES TO "DECOMPOSE" TOTAL ANGULAR MOMENTUM

$$\left|\tfrac{3}{2}\ -1/2\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle + \sqrt{1/2}\,|1\,-1\rangle\left|\tfrac{1}{2}\ +\tfrac{1}{2}\right\rangle$$

PROBABILITY OF POSSIBLE FINAL STATES IN DECAY

$A \to B + C$

$$\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle \to\ ?\,|1\ \ ?\rangle + ?\left|\tfrac{1}{2}\ \ ?\right\rangle$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

SPIN AND STATISTICS

- INTRINSIC ANGULAR MOMENTUM IS AN ATTRIBUTE
- ALL ELECTRONS HAVE SAME MASS, + SAME SPIN

- PARTICLES CAN HAVE EITHER
  - INTEGRAL SPIN $\gamma, W^{\pm}, Z, g$
  - $\frac{1}{2}$ INTEGRAL QUARK, LEPTON, PROTON ...

INTEGRAL SPIN, $\frac{1}{2}$ INTEGRAL SPIN — VERY DIFFERENT

TWO INDENTICAL PARTICLES, OVERALL WAVE FUNCTION

$$|\psi(1,2)\rangle = |\bar{x}_1, J_{z1}, \bar{x}_2, J_{z2}\rangle$$

THERE EXISTS AN INTERCHANGE OPERATOR

EXCHANGES POSITION OF PARTICLES

$$I_{OP}\ |1,2\rangle \longrightarrow |2,1\rangle$$

$$\mathrm{IOP}\,|1,2\rangle \rightarrow |2,1\rangle$$

INTEGER SPIN

BOSE-EINSTEIN

$$|1,2\rangle = +|2,1\rangle$$

- SYMMETRIC
- NO OBSERVABLE DIFFERENCE UNDER EXCHANGE (IDENTICAL!)
- CAN HAVE MANY BOSONS IN SAME QUANTUM STATE → LIGHT (γ^ν)

$\frac{1}{2}$ INTEGER SPIN

FERM-DIRAC

$$|1,2\rangle = -|2,1\rangle$$

- ANTI SYMMETIC
- BOTH WAVE FUNCTIONS DESCRIBE SAME STATE (IDENTICAL!)

ONLY POSSIBLE IF

$$|1,2\rangle = |2,1\rangle = 0$$

- NO TWO FERMIONS CAN BE IN SAME QUANTUM STATE → ATOMS