

FORM FACTORS - SIZE & SHAPE OF NUCLEI & PARTICLES

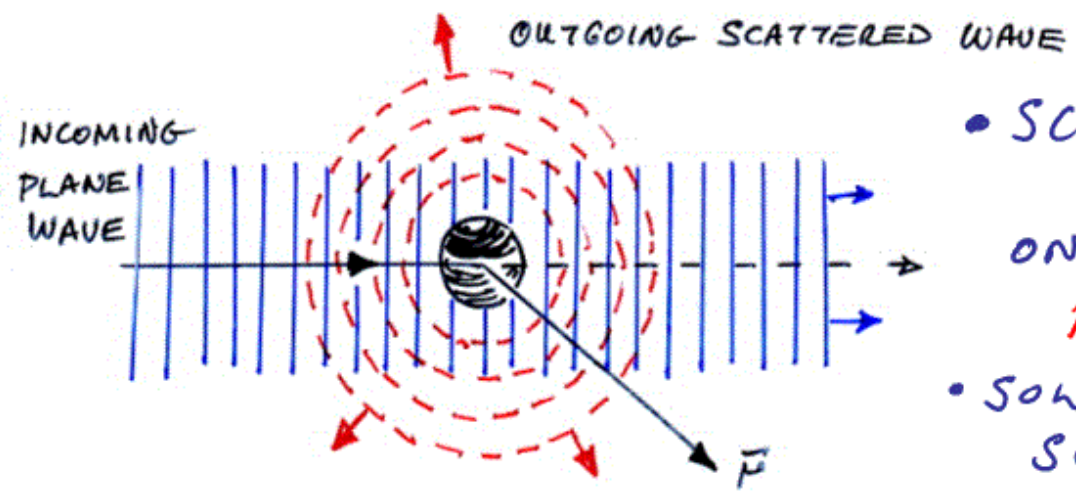
SCATTERING EXPERIMENTS

- DETERMINE NUCLEAR SIZE
- DETERMINE NUCLEAR SHAPE
 - DISTRIBUTION OF "MATTER"
 - DISTRIBUTION OF ELECTRIC CHARGE
- DETERMINE SIZE OF PROTON
- DETERMINE SHAPE OF PROTON
 - DISTRIBUTION OF ELECTRIC CHARGE

↳ DIRECT OBSERVATION
OF QUARKS

SCATTERING IN NON-RELATIVISTIC QUANTUM MECHANICS

- WE WANT TO GET A FEEL FOR HOW EXPERIMENTS CAN DETERMINE STRUCTURE
- BUILD ON YOUR KNOWLEDGE OF QUANTUM MECHANICS



- SCATTERING IS A WEAK PERTURBATION ON FREELY PROPAGATING PLANE WAVE STATE
- SOLVE TIME DEPENDENT SCHRÖDINGER

$$(\nabla^2 + k^2)\psi = \frac{2m}{\hbar^2} V\psi$$

• INCOMING $\psi = e^{i\vec{p}\cdot\vec{r}/\hbar}$

• OUTGOING $\psi = e^{i\vec{p}\cdot\vec{r}/\hbar} + f(\theta, \phi) \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{|\vec{r}|}$

OUTGOING PLANE WAVE

OUTGOING SPHERICAL WAVE

SOLUTION BY GOLDEN RULE (FERMI)

TRANSITION $|\psi_a\rangle \rightarrow |\psi_b\rangle$ CAUSED BY POTENTIAL V


$$\frac{d\sigma}{d\Omega_b} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\langle \psi_b | V | \psi_a \rangle|^2$$

TAKE $|\psi_a\rangle = \exp(i\bar{k}_a \cdot \bar{r})$ $\bar{p} = \hbar \bar{k}$

WHAT ABOUT $|\psi_b\rangle$? SAY V IS SUCH A SMALL PERTURBATION, THAT $|\psi_b\rangle \approx |\psi_a\rangle$

$$|\psi_a\rangle = \exp(i\bar{k}_a \cdot \bar{r}) \rightarrow |\psi_b\rangle = \exp(i\bar{k}_b \cdot \bar{r})$$

$$\begin{aligned} \frac{d\sigma}{d\Omega_b} &= \left(\frac{m}{2\pi\hbar^2}\right)^2 |\langle \psi_b | V | \psi_a \rangle|^2 = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int \psi_b^* V \psi_a \right|^2 \\ &= \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int V e^{i(\bar{k}_a - \bar{k}_b) \cdot \bar{r}} \right|^2 \end{aligned}$$

NORMALIZATION
& DENSITY OF
FINAL STATES 

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int V e^{i(\vec{k}_a - \vec{k}_b) \cdot \vec{r}} \right|^2$$

INTERACTION POTENTIAL

PHASE CHANGE

MOMENTUM TRANSFER

GENERALLY

$$\frac{d\sigma}{d\Omega} = |f(\vec{q})|^2 \quad \vec{q} = \vec{p} - \vec{p}'$$

FOR OUR SCATTERING EXAMPLE

$$f(\vec{q}) = \frac{-m}{2\pi\hbar^2} \int V(\vec{x}) e^{i\vec{q} \cdot \vec{x}/\hbar} d^3x$$

SCATTERING POTENTIAL

PLANE WAVE STATE

MOMENTUM TRANSFER

FOR A SPHERICALLY SYMMETRIC POTENTIAL (eg. COULOMB)

$$f(\vec{q}) = \frac{-m}{2\pi\hbar^2} \int V(\vec{x}) e^{i\vec{q}\cdot\vec{x}/\hbar} d^3x$$

∫ OVER $4\pi \rightarrow$ CHANGE VARIABLE TO $z = i q x \cos\theta / \hbar$
 $x = |\vec{x}|$ $q = |\vec{q}|$

$$f(q^2) = \frac{-2m}{\hbar q} \int_0^\infty dx x \sin\left(\frac{q x}{\hbar}\right) V(x)$$

TO MAKE THINGS CONCRETE NEED THEORETICAL MODEL OF $V(x)$

TAKE COULOMB INTERACTION \rightarrow RUTHERFORD SCATTERING

$$V(x) = \frac{q_1 q_2}{x} = \frac{Z Z e^2}{x}$$

$Z =$ BEAM CHARGE

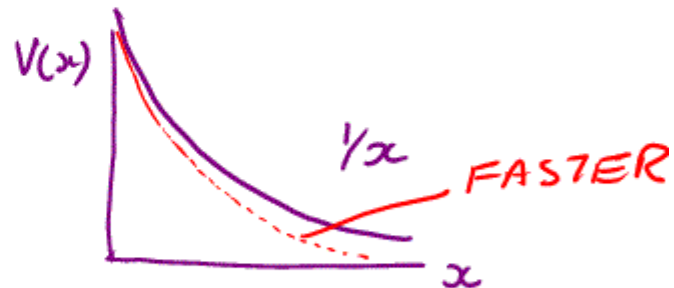
$Z =$ TARGET CHARGE

$$f(q^2) = \frac{-2m}{\hbar q} \int_0^{\infty} dx \cdot x \sin\left(\frac{qx}{\hbar}\right) \cdot \frac{3Ze^2}{x}$$

IS, UNFORTUNATELY, DIVERGENT.

ASSUME NUCLEAR TARGET SHIELDED BY
ATOMIC ELECTRONS

$$V(x) = \frac{3Ze^2}{x} \exp\left(-\frac{x}{a}\right)$$



$$f(q^2) = \frac{2m 3Ze^2}{q^2 + \frac{\hbar^2}{a^2}}$$

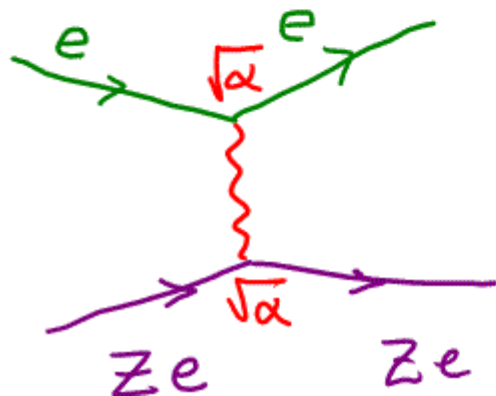
PARAMETER
~ DIMENSIONS
OF ATOM

$$f(q^2) = \frac{2m_3 Z e^2}{q^2 + \left(\frac{\hbar}{a}\right)^2}$$

WE CAN CHOOSE q^2 SUCH THAT THIS TERM NEGLIGIBLE

$$f(q^2) = \frac{2m_3 Z e^2}{q^2} \rightarrow \left(\frac{d\sigma}{d\Omega} \right)_{\text{RUTH}} = \frac{4m^2 (Z e^2)^2}{q^4}$$

NOTE THAT FEYNMAN RULES GIVE



$$\rightarrow f(q^2) = \text{CONSTANT} \cdot \frac{Z e^2}{q^2}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RUTM} = \frac{4m^2 (Zze^2)^2}{q^4}$$

GO BACK TO LECTURE #3

$$q = 2p \sin \theta/2$$

ASSUME NON RELATIVISTIC

$$T = \frac{1}{2} m v^2 \rightarrow T^2 = \frac{p^4}{4m^2} = \frac{q^4}{4m^2 16 \sin^4 \theta/2}$$

$$\frac{d\sigma}{d\Omega} = \frac{(Zze^2)^2}{T^2 16 \sin^4 \theta/2}$$

OR

$$\frac{d\sigma}{d\Omega} = \frac{4m^2 (Zze^2)^2}{p^4 \sin^4 \theta/2}$$

RUTHERFORD DIFFERENTIAL CROSS SECTION IS!

A THEORETICAL MODEL

VALIDITY DEPENDS ON VALIDITY OF ASSUMPTIONS

- COULOMB POTENTIAL - WEAK PERTURBATION
- ∞ HEAVY TARGET - NO RECOIL
- SPIND PROJECTILE & TARGET
- NO STRUCTURE, OR SPATIAL EXTENT OF TARGET OR BEAM PROJECTILE

DEVIATIONS FROM RUTHERFORD GIVE INFORMATION ON!

- DIFFERENT POTENTIAL ACTING
- NON POINT PARTICLES
- INTERNAL STRUCTURE
- SPIN OF PARTICLES

MOTT DIFFERENTIAL CROSS SECTION

ELECTRON BEAMS - RELATIVISTIC, SPIN $1/2$
(E.G. SLAC LINAC) - NEED A GENERALIZATION
OF RUTHERFORD - RELATIVISTIC SPIN $1/2$ BEAM

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{MOTT}} = 4(Ze^2)^2 \frac{E^2}{(q/c)^4} \left(1 - \underbrace{\beta^2 \sin^2 \frac{\theta}{2}}\right)$$

INCOMING ELECTRON
ENERGY

RELATIVISTIC EFFECT OF
ELECTRON SPIN HAVING
MAGNETIC INTERACTION WITH TARGET

$\rightarrow 0$ AS $\beta \rightarrow 0$; $E \rightarrow mc^2$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{MOTT}} \rightarrow 4(Ze^2)^2 \frac{m^2 c^4}{q^4 c^4} = 4(Ze^2)^2 \frac{m^2}{q^4} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{RUTH}}$$

RUTHERFORD & MOTT IGNORE SPATIAL EXTENT
OR INTERNAL STRUCTURE
OF TARGET

HOWEVER, WE DO EXPERIMENTS TO DETERMINE
EXPERIMENTALLY:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{EXPERIMENT}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{MOTT}} \cdot |F(q^2)|^2$$

MEASURE



EXPERIMENT

CALCULABLE
POINT SCATTERING



THEORY

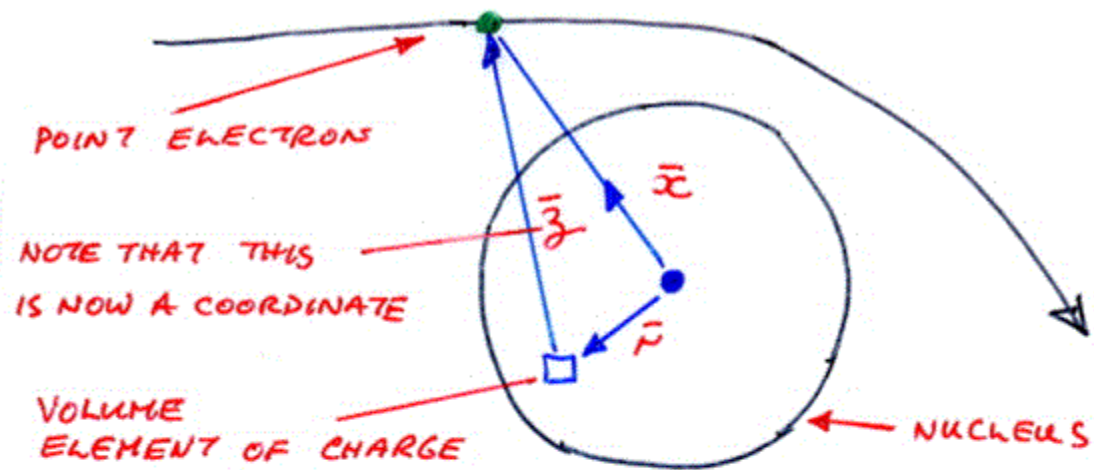
FORM FACTOR

INCORPORATES
SPATIAL EXTENT

Eg. DISTRIBUTION
OF ELECTRIC
CHARGE

INFER

RUTHERFORD SCATTERING FROM EXTENDED TARGET



CHARGE DENSITY IN NUCLEUS $\rho(r)$

IN VOLUME ELEMENT $d^3r = r^2 \sin\theta \, d\theta \, d\phi \, dr$

CHARGE IS $Ze \rho(r) d^3r$ CONTRIBUTION OF THIS ELEMENT TO COULOMB POTENTIAL AT ELECTRON

$$dV(x) = \frac{-Ze^2}{r} \exp\left(\frac{-Z}{a}\right) \rho(r) dr$$

COULOMB $\frac{-Ze^2}{r}$ SHIELDING $\exp\left(\frac{-Z}{a}\right)$ CHARGE DENSITY $\rho(r)$

$$dV(x) = -\frac{Ze^2}{r} \exp\left(-\frac{r}{a}\right) \rho(r) dr$$

POTENTIAL AT x DUE TO WHOLE NUCLEUS

$$V(x) = -Ze^2 \int d^3r \frac{\rho(r)}{r} \exp\left(-\frac{r}{a}\right)$$

PUT THIS POTENTIAL INTO BORN APPROXIMATION

$$f(q^2) = \frac{-m}{2\pi\hbar^2} \int V(x) e^{-i\vec{q}\cdot x/\hbar} d^3x$$

NEW COORDINATE SYSTEM $\vec{x} = \vec{r} + \vec{z}$

$$f(q^2) = \frac{mZe^2}{2\pi\hbar^2} \int d^3r e^{i\vec{q}\cdot\vec{r}/\hbar} \rho(r) \int d^3z \frac{e^{-z/a}}{z} e^{i\vec{q}\cdot\vec{z}/\hbar}$$

$$f(q^2) = \frac{mZe^2}{2\pi\hbar^2} \int d^3r e^{i\vec{q}\cdot\vec{r}} \rho(r) \int d^3r_2 \frac{e^{-2/a}}{r_2} e^{i\vec{q}\cdot\vec{r}_2/\hbar}$$

INTEGRAL OVER VOLUME OF NUCLEUS

INTEGRAL OVER ELECTRON ORBIT OF COULOMB INTERACTIONS

FORM FACTOR OF NUCLEUS

$$= \frac{4\pi\hbar^2}{q^2 + \left(\frac{\hbar^2}{a}\right)^2} \rightarrow \frac{4\pi\hbar^2}{q^2}$$

$$F(q^2) = \int d^3r e^{i\vec{q}\cdot\vec{r}/\hbar} \rho(r)$$

FOURIER TRANSFORM OF SPATIAL CHARGE DENSITY
CHARGE DENSITY IN MOMENTUM SPACE

$\sim \frac{1}{q^2} \rightarrow$ MOMENTUM TRANSFER

$$\left(\frac{d\sigma}{d\Omega}\right) = |f(q^2)|^2 = |F(q^2)|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{POINT TARGET}}$$

USE DIFFERENT BEAMS TO PROBE DIFFERENT ASPECTS OF SUBATOMIC MATTER

SIMPLEST IF BEAM IS A POINT PARTICLE

ELECTRONS — PROBE ELECTRIC CHARGE DISTRIBUTION

NEUTRINOS — PROBE DISTRIBUTION OF WEAK CHARGE

HADRONS — PROBE DISTRIBUTION OF STRONGLY INTERACTING MATTER
(BOUND STATES OF QUARKS)

↑
NOT POINT LIKE BEAM
SO HAVE CONVOLUTION OF BEAM
AND TARGET FORM FACTORS

FROM FORM FACTOR GET ANOTHER INSIGHT
 INTO WHAT BEAM MOMENTUM IS NEEDED TO
 RESOLVE STRUCTURE IN TARGET

$$F(q^2) = \int d^3r e^{i\vec{q}\cdot\vec{r}/\hbar} \rho(r)$$

IF ~ 1 THEN $F(q^2) =$ CHARGE ON NUCLEUS

FOR $F(q^2)$ TO BE DIFFERENT
 FROM POINT SCATTERING

POINT SCATTERING

$$e^{i\vec{q}\cdot\vec{r}/\hbar} < 1 \rightarrow \frac{\vec{q}\cdot\vec{r}}{\hbar} \sim 1$$

SO $\vec{q} \sim \hbar/r \rightarrow q > 20 \text{ MeV}/c$

$$197 \frac{\text{MeV}}{c} \cdot \text{fm}$$

10 fm FOR NUCLEAR RADIUS

EXPERIMENTAL DETERMINATION OF TARGET STRUCTURE

MEASURE FORM FACTOR $\left(\frac{d\sigma}{d\Omega}\right)_{\text{EXPT}} = |F(q^2)|^2 \frac{d\sigma}{d\Omega}$ POINT
↓
THEORY

DO A FOURIER TRANSFORM TO GET
CHARGE DENSITY IN REAL SPACE

$$\rho(r) = \frac{1}{(2\pi)^3} \int d^3q F(q^2) e^{-i\vec{q} \cdot \vec{r}/\hbar}$$

HAVE TO MEASURE $\overbrace{\hspace{10em}}^{\text{FORM FACTOR}}$ OVER ALL q^2 - DIFFICULT.

USUALLY HYPOTHESIZE FORM OF $\rho(r)$

PUT INTO $F(q^2) = \int d^3r e^{i\vec{q} \cdot \vec{r}/\hbar} \rho(r)$ AND

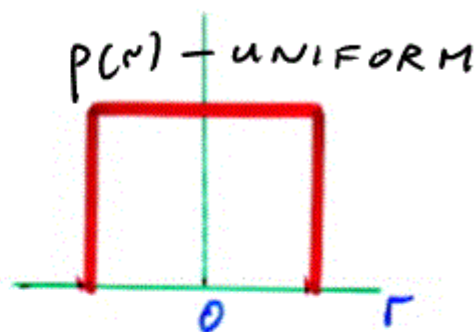
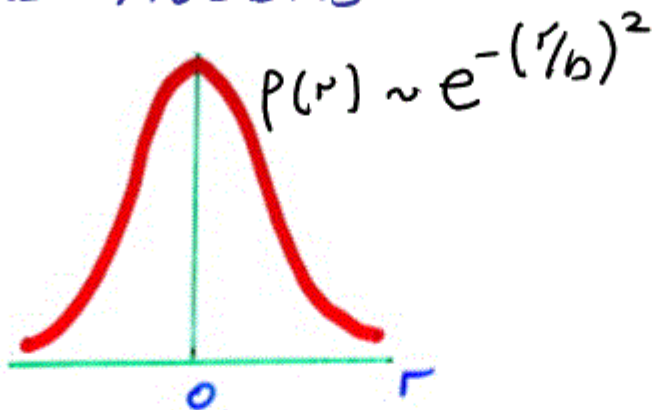
DETERMINE WHICH $\rho(r)$ GIVES BEST FIT TO
THE EXPERIMENTAL $F(q^2)$

ELECTRIC CHARGE DISTRIBUTIONS IN NUCLEI

TWO SIMPLE MODELS

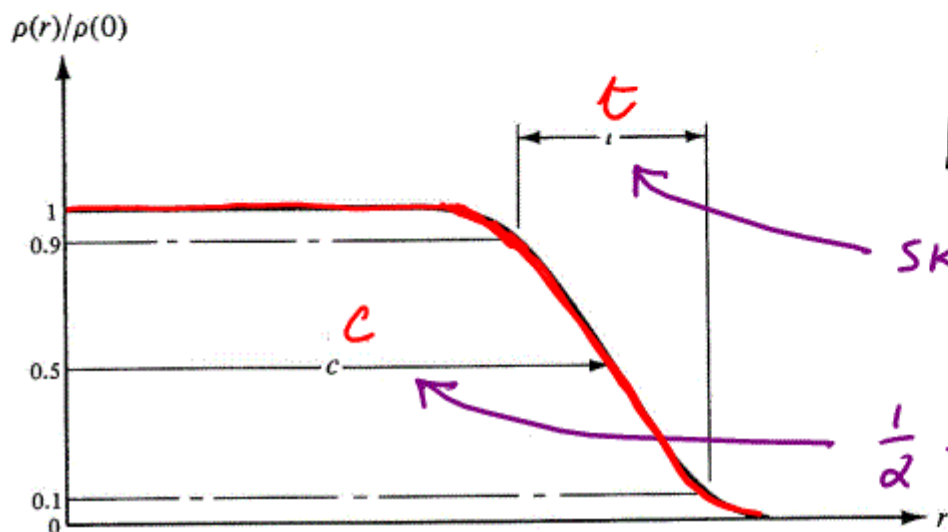
GAUSSIAN

$$F(q^2) = \exp\left[\frac{-q^2 b^2}{4\pi^2}\right]$$



$F(q^2)$ COMPLEX

STANDARD NUCLEAR MODEL

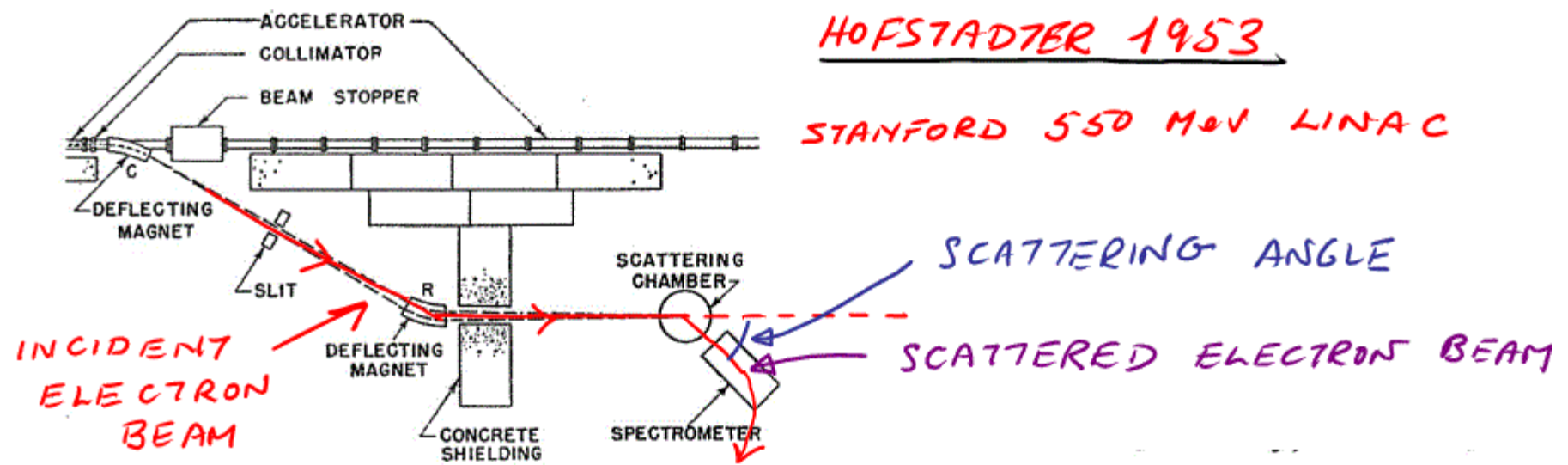


$$\rho(r) = \frac{1}{1 + e^{(r-c)/a}}$$

SKIN THICKNESS $t = (4 \ln 3) a$

HOFSTADTER 1953

STANFORD 550 MeV LINAC

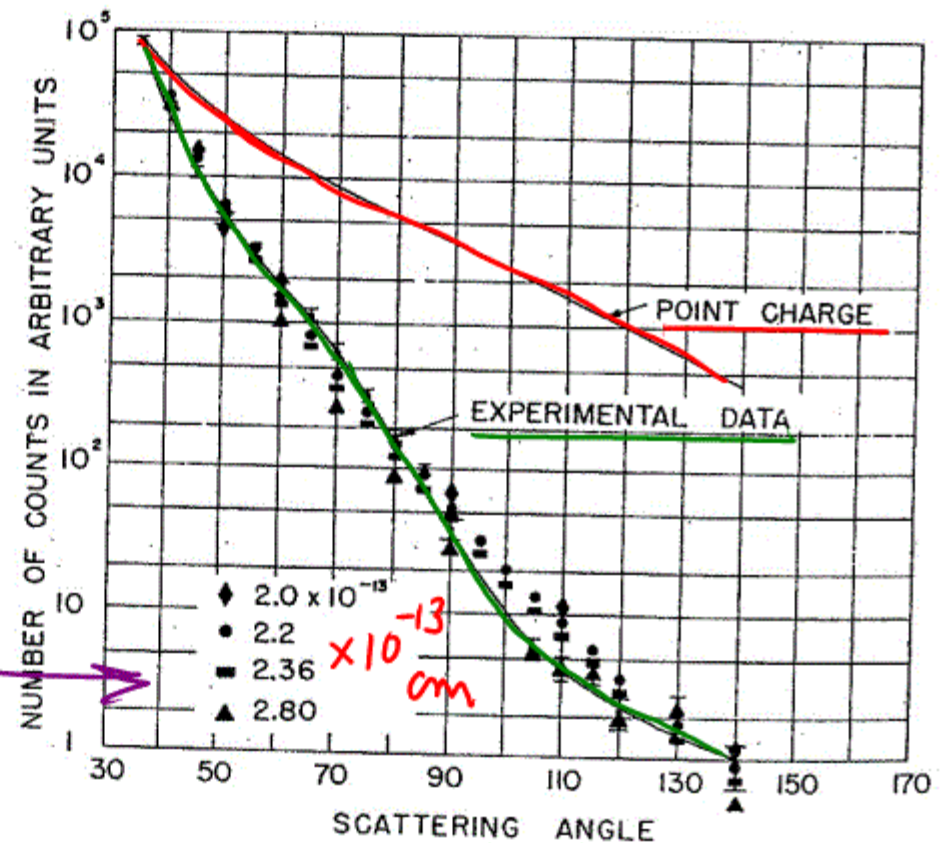


NUCLEAR SIZE

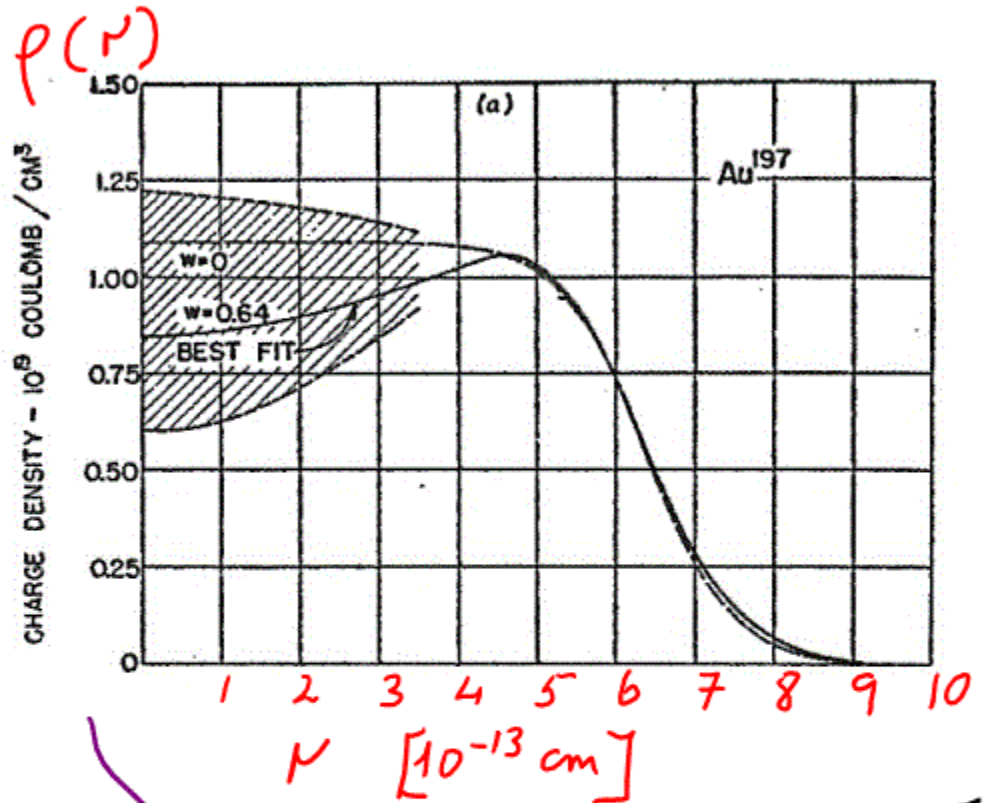
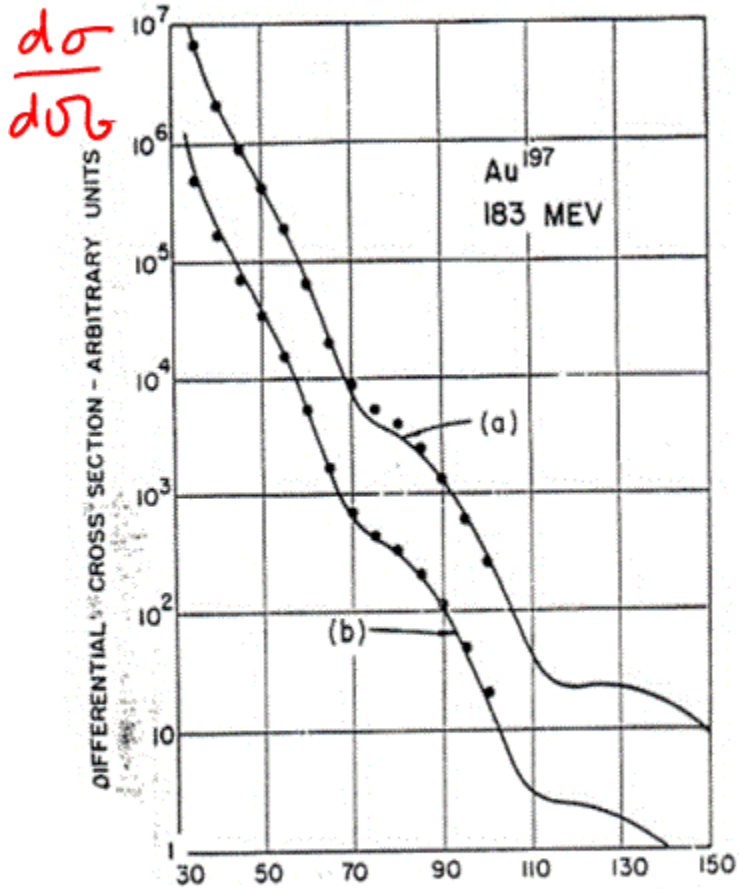
$eAu \rightarrow eAu$

NUCLEUS NOT A POINT PARTICLE

NUCLEAR RADIUS
HYPOTHESIZED IN
GAUSSIAN FORM
FACTOR



DETERMINATION OF CHARGE DISTRIBUTION IN NUCLEI

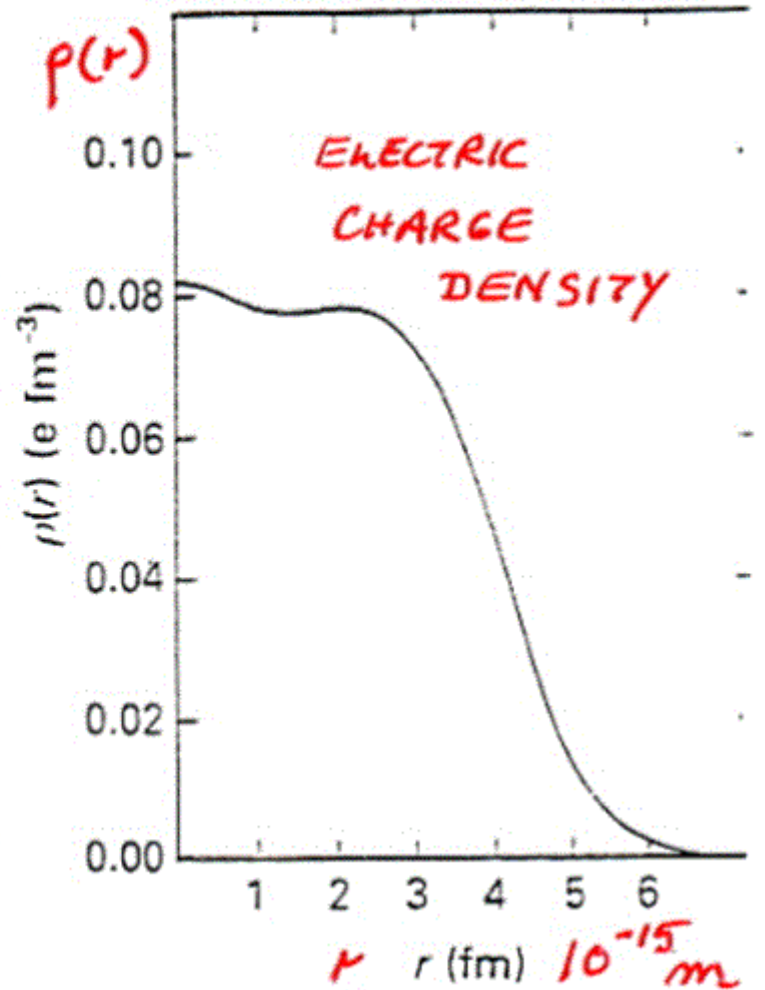
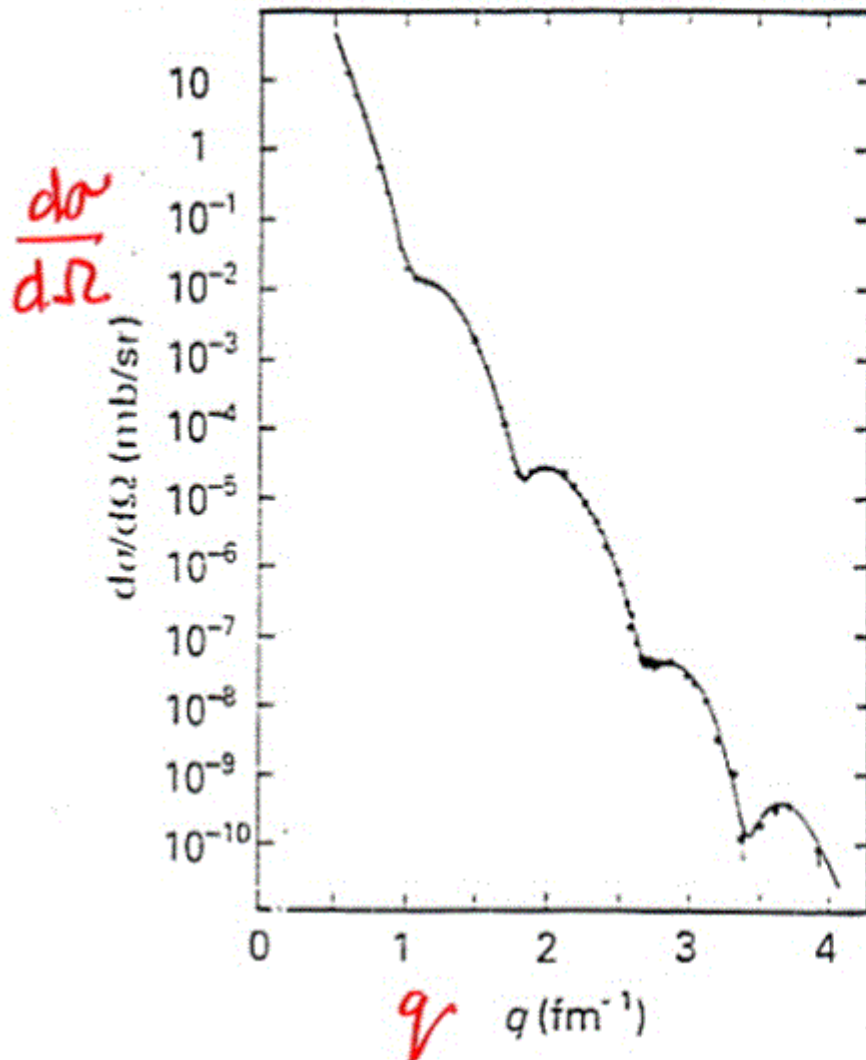


$$\frac{d\sigma}{d\nu b} = \left(\frac{d\sigma}{d\nu b} \right)_{\text{MOTT}} |F(q^2)|^2$$

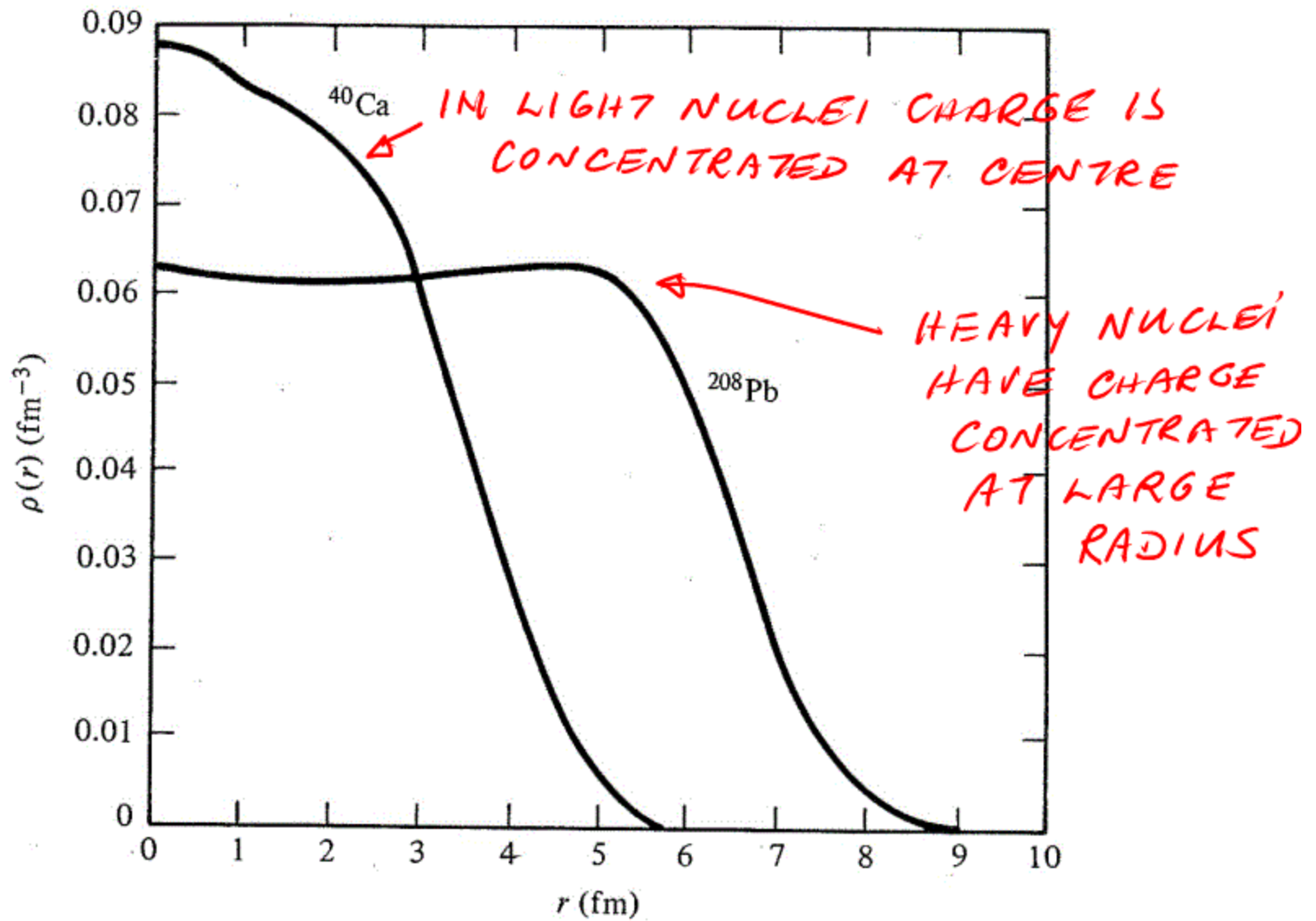
(POINT)

$$P(r) = \frac{1}{(2\pi)^3} \int d^3q F(q^2) e^{-i\vec{q} \cdot \vec{r} / \hbar}$$

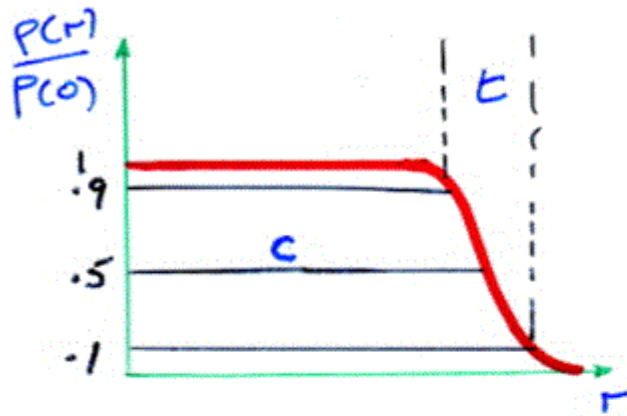
450 MeV ELECTRONS ON $^{58}_{28}\text{Ni}$



COMPARISON OF LIGHT & HEAVY NUCLEI



ELECTRIC CHARGE DISTRIBUTION IN NUCLEI



$$\rho(r) = \frac{\rho_0}{1 + e^{(r-c)/a}}$$

$$t = (4 \ln 3) a$$

REPRESENT AN OBJECT WITH DIFFUSE BOUNDARY BY

ROOT MEAN SQUARE CHARGE RADIUS $\langle r^2 \rangle^{1/2}$

$$\langle r^2 \rangle = \int d^3r r^2 \rho(r)$$

THESE EXPERIMENTS SHOW

$$\langle r^2 \rangle^{1/2} = r_0 A^{1/3}$$

0.94 fm

← MASS NUMBER
= SUM OF PROTONS
AND NEUTRONS

$$\langle r^2 \rangle^{\frac{1}{2}} = r_0 A^{1/3}$$

• VOLUME \propto (RADIUS)³ \rightarrow NUCLEAR VOLUME $\propto (A^{1/3})^3 \propto A$

• NUCLEAR MASS CLEARLY $\propto A$

\therefore DENSITY OF NUCLEAR MATTER CONSTANT
FOR ALL NUCLEI (EXCEPT LIGHTEST)

• ELECTRON EXPERIMENTS

$$C = 1.18 A^{-1/3} - 0.48 \text{ fm}$$

$$t = 2.4 \text{ fm}$$

$$\rightarrow \rho_{\text{NUCLEUS}} \approx 0.17 \text{ NUCLEONS / fm}^3$$