Physics 3575 Problem Set #1 Solutions

Problem #1

i) Minkowski Notation: time (or space) is imaginary.

$x^\mu = (ict, x, y, z)$ so that $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ ①

Minkowski Metric: rather than imaginary time, introduce a metric where

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ so that $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ ②

We can see that ① = (-1)(②), so that we only have to show Lorentz Invariance for one case, and the other follows. This implies that our choice of metric (providing it describes our space-time) does not matter!

Now consider a Lorentz boost in $\vec{x}$:

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ so that

$t' = \gamma(t - vx/c^2)$

$x' = \gamma(x - vt)$

$y' = y$

$z' = z$

(note: boosts in y and z follow similarly. We ignore other types of Lorentz transformations).

Then,

$$\begin{aligned}
ds^2 &= -c^2\gamma^2\left(dt - \frac{v\,dx}{c^2}\right)^2 + \gamma^2(dx - v\,dt)^2 + \dots \\
&= -c^2\gamma^2\left(dt^2 - 2v\,dx\,dt/c^2 + v^2dx^2/c^4\right) + \gamma^2\left(dx^2 - 2v\,dx\,dt + v^2dt^2\right) + \dots \\
&= \gamma^2\left(-c^2 dt + 2v\,dx\,dt - v^2dx^2/c^2 + dx^2 - 2v\,dx\,dt + v^2dt^2\right) + \dots \\
&= \gamma^2\left(-c^2(1 - v^2/c^2)\,dt^2 + (1 - v^2/c^2)\,dx^2\right) + \dots \\
&= \gamma^2\left(-c^2\gamma^{-2}dt^2 + \gamma^{-2}dx^2\right) + \dots \\
&= -c^2dt^2 + dx^2 + dy^2 + dz^2 \quad \checkmark
\end{aligned}$$

The invariant interval in energy-momentum space is:

$$p^2 = (E, \vec{p}) \cdot (E, \vec{p})$$
$$= E^2 - \vec{p} \cdot \vec{p}$$
$$= m^2 \rightarrow \text{rest mass.}$$

ii) For a space-like interval, $|c\Delta t| < |\Delta x|$
If we choose $\Delta t' = 0$, then $|c(0)| < |\Delta x|$, which does not violate our inequality, so such a space-like interval can be found.
Conversly, if we have $\Delta x' = 0$, looking at the Lorentz Transformation we find: $\Delta x' = \gamma(\Delta x - \beta c \Delta t)$

$$0 = (\Delta x - \beta c \Delta t)$$
$$\Rightarrow \beta = c\Delta t / \Delta x$$

The maximum β we can have is $1 (\beta = v/c)$ so $1 > c\Delta t / \Delta x$
Rearranging, we get $\Delta x / \Delta t > c$.
So, for $\Delta x' = 0$ for 2 spacelike events, the velocity must be greater than c. $\therefore$ no such frame exists.

iii) For a time-like interval, $|c\Delta t| > |\Delta x|$.
If we choose $\Delta x' = 0$, then $|c\Delta t'| > |(0)|$, so such a time-like interval can be found.
Conversely, for $\Delta t' = 0$, looking at the Lorentz transformation again, $\Delta t' = \gamma(\Delta t - v\Delta x / c^2)$

$$0 = \Delta t - v/c^2 \, \Delta x$$
$$\Rightarrow v = c^2 \, \Delta t / \Delta x$$

But the maximum velocity is c, so $v < c$ and $c^2 \Delta t / \Delta x < c$.
Rearranging, we get $\Delta x / \Delta t > c$; impossible.

iv) This special case is when two events occur at the same time <u>and</u> same place ie, $\Delta t = 0$ and $\Delta x = 0$

Physics 357S - Problem Set #1 Solutions

Problem #2

First diagram: $q^2 = (p_1 - p_2)^2$

$= p_1^2 + p_2^2 - 2p_1p_2$

$= m_1^2 + m_2^2 - 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$

$= m_1^2 + m_2^2 - 2E_1E_2 - 2p_1p_2\cos\theta$

note $2E_1E_1 > m_1^2 + m_2^2$ always

$2p_1p_2\cos\theta < 2E_1E_2$ always $\rightsquigarrow q^2$ -ve

For $E \gg m$, $q^2 = -2E_1E_2 - 2p_1p_2\cos\theta$

$= -2E_1E_2(1+\cos\theta) \Rightarrow 2 \geq (1+\cos\theta) \geq 0 \quad \therefore q^2 < 0$

Second diagram for $E \gg m$, $q^2 = (p_1 + p_2)^2$

$= 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$

$= 2E_1E_2 - 2E_1E_2\cos\theta$

$= 2E_1E_2(1-\cos\theta) \Rightarrow 2 \geq (1-\cos\theta) \geq 0 \quad \therefore q^2 > 0$

OR $(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1p_2$

$= m_1^2 + m_2^2 + 2E_1E_2 - 2|p_1||p_2|\cos\theta$

for colliding beams — annihilation $\cos\theta = -1$

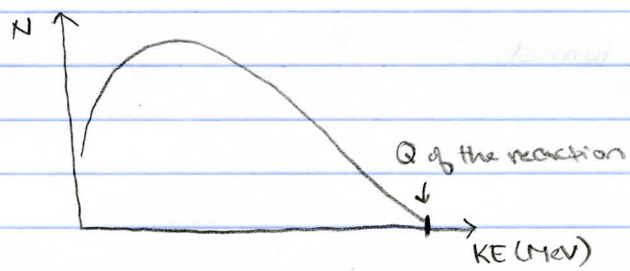
$= m_1^2 + m_2^2 + 2E_1E_2 + 2|p_1||p_2|$

+ve

Physics 357S - Problem Set #1 Solutions

Problem #3

i) Pauli proposed the existence of the neutrino due to an interesting feature of β-decay. Basically, what was observed was a spectrum of measured electron energies, with many of the electrons having energies significantly lower than expected. This continuous spectrum could be explained if there was another invisible particle involved, which carries away some of the energy. The shape of the electron energy spectrum could then be predicted.



ii) $p_{K^+} = 1000\ GeV/c$

$$\gamma = \sqrt{1-(v/c)^2} = \sqrt{1+(p/m_0c)^2}$$
$$= sqrt(1 + (1000\,GeV/493.7\,MeV)^2)$$
$$= 2025.52$$

$$\tau_{lab} = \gamma\, \tau_0$$
$$= (2025.52)(1.24\times10^{-8}\,s)$$
$$= 2.51\times10^{-5}\,s$$

$$t = d/v$$
$$= d/(p/\gamma m_0)$$
$$= 650\,m/(1000\,GeV/2025.52\cdot 493.7\,MeV)$$
$$\cong 650\,m/c = 2.168\times10^{-6}$$

$$N/N_0 = exp(-t/\tau_{lab})$$
$$= exp(-2.168\times10^{-6}/2.51\times10^{-5}\,s)$$
$$= 91.7\,\%$$

∴ 8.3% decay in the tunnel.

$p_{\pi^+} = 1 \text{ MeV}/c$

$\gamma = \sqrt{1 + (P/m)^2}$
$= \text{sqrt}(1 + (1\text{MeV}/139.6\text{MeV})^2)$
$\cong 1$

$\therefore \tau_{lab} \cong \tau_0 = 2.6 \times 10^{-8}\text{s}$

$v \cong P/m_0 = \frac{1\text{MeV}}{139.6\text{MeV}} = 0.0072c$
$\Rightarrow t = d/v = 3.03 \times 10^{-4}\text{s}$

$\frac{N}{N_0} = \exp(-t/\tau_0)$
$= \exp(-3.03 \times 10^{-4}/2.6 \times 10^{-8})$
$\cong 0\%$

$\therefore$ 100% decay in the tunnel.

$p_{\pi^+} = 1 \text{ MeV}/c$

$$\gamma = \sqrt{1 + (p/m)^2}$$
$$= \text{sqrt}(1 + (1\text{MeV}/139.6\text{MeV})^2)$$
$$\cong 1$$

$\therefore \tau_{lab} \cong \tau_0 = 2.6\times10^{-8}\,s$

$v \cong p/m_0 = \frac{1\text{MeV}}{139.6\text{MeV}} = 0.0072c$

$\Rightarrow t = d/v = 3.03\times10^{-4}\,s$

$$\frac{N}{N_0} = exp(-t/\tau_0)$$
$$= exp(-3.03\times10^{-4}/2.6\times10^{-8})$$
$$\cong 0\ \%$$

$\therefore$ 100% decay in the tunnel.

Physics 357S Problem Set #1 Solutions

Problem ~~#3~~ 4

i) $\frac{dN}{dt} = -\omega_1 N - \omega_2 N$

$dN = (-\omega_1 N - \omega_2 N)dt$

$dN/N = (-\omega_1 - \omega_2)\, dt$

$N = N_0 \exp[-(\omega_1 + \omega_2)t]$

ii) In general, decay formula is $N = N_0 \exp[-\omega t]$ $\therefore$ it is evident that $\omega_T = \omega_1 + \omega_2 = \omega$

iii) In general, we can also write $N = N_0 \exp[-t/\tau]$ where τ = mean lifetime. $\therefore$ $\tau = \omega^{-1} = (\omega_1 + \omega_2)^{-1}$

iv) $f_1 = \omega_1/\omega$ Extending to n: $f_i = \omega_i/\omega$ where $\omega = \sum_{j=1}^{n} \omega_j$

$f_2 = \omega_2/\omega$

✓

v) $\omega = \tau^{-1} = (1.24\times10^{-8}\, s)^{-1} = 8.08\times10^{7}$ ✓

$\therefore \omega_1 = f_1\omega = (0.635)(8.08\times10^7) \cong 5.1\times10^7$ ✓

$(0.212)(\quad " \quad) \cong 1.7\times10^7$

$(0.056)(\quad " \quad) \cong 4.5\times10^6$ etc

$(0.017)(\quad " \quad) \cong 1.3\times10^6$

$(0.032)(\quad " \quad) \cong 2.6\times10^6$

$(0.048)(\quad " \quad) \cong 3.9\times10^6$

Some examples of "systematics":

- charge is conserved
- K changes into pion electrons or muons
- there is always a neutrino when there is an electron or muon

Physics 357S Problem Set #1 Solutions

Problem #5

i) $dN_1/dt = -\omega_1 N_1(t)$ ①

$dN_2/dt = -\omega_2 N_2(t) + \omega_1 N_1(t)$ ②

From ①, $N_1(t) = N_{10}\exp(-\omega_1 t)$

Subbing ① into ②,

$dN_2 = [-\omega_2 N_2(t) + \omega_1 N_{10}\exp(-\omega_1 t)]\,dt$

$\therefore N_2 = \left(\frac{N_{10}\omega_1}{\omega_2-\omega_1}\right)\left(e^{-\omega_1 t} - e^{-\omega_2 t}\right)$

Then, $\omega_1 = \left(\frac{1}{7.2\,\text{days}}\right)$, $\omega_2 = \left(\frac{1}{200\,\text{days}}\right)$, differentiating N_2 gives:

$dN_2/dt = \frac{N_{10}\omega_1}{(\omega_2-\omega_1)}\left(-\omega_1 e^{-\omega_1 t} + \omega_2 e^{-\omega_2 t}\right)$

$0 = \frac{N_{10}\omega_1}{\omega_2-\omega_1}\left(-\omega_1 e^{-\omega_1 t} + \omega_2 e^{-\omega_2 t}\right)$

$\Rightarrow \omega_1 e^{-\omega_1 t} = \omega_2 e^{-\omega_2 t}$

$\omega_1/\omega_2 = e^{-\omega_2 t}\, e^{\omega_1 t}$

$\ln(\omega_1/\omega_2) = -\omega_2 t + \omega_1 t$

$t = \ln(\omega_1/\omega_2)(\omega_1-\omega_2)^{-1}$

$t = 24.8$ days

ii) Here, a range of answers were acceptable. Basically, you had to realize that you can calculate the number of fissions by dividing the total energy released by the reactor (3 GW × 1 yr) by the energy released per fission (200 MeV).

The other "trick" was to recognize that the # of Cs atoms is <u>NOT</u> the same as the activity of the Cs – you had to figure out how many Cs atoms decayed. This is done similarly as in part i).

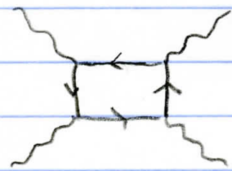
Some other comments:

- Chernobyl was dangerous, so if your activity was very small then perhaps you did the question wrong!
- If you are asked to compare the radiation released from Fukushima and Chernobyl, I expect you to quote some actual numbers!
- $1000000 \text{ km}^2 = 10^{12} \text{ m}^2$ (not $(10^6 \text{ km}^2)^2$!)
- Bq is disintigrations per second
- your answer should be on the order of $\sim 10^4$ Bq/m^2

Physics 357S – Problem Set #1 Solutions

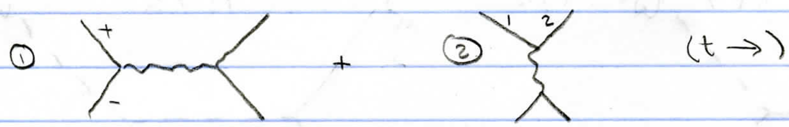
Problem #6:

i) Delbruck scattering:



This is a result of photons scattering off of a Coulomb field.

ii) Bhabha scattering:



① $q^2 = (p_1 + p_2)^2$

$m_\gamma^2 = (E_+ + E_-)^2 - (\vec{p}_+ + \vec{p}_-)^2$

$m_\gamma^2 = (E_+ + E_-)^2 - \vec{O}$ (ie., $\Sigma \vec{p} = 0$, head-on collision)

$m_\gamma = 2E$

$m_\gamma = 200\,GeV$

$v_\gamma = 0\,m/s$ from conservation of momentum.

② $q^2 = (p_1 - p_2)^2$

$m_\gamma^2 = p_1^2 + p_2^2 - 2p_1p_2$

$m_\gamma^2 = m_1^2 + m_2^2 - 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$

$m_\gamma^2 = -2E_1E_2 + 2\vec{p}_1 \cdot \vec{p}_2$ $\quad (E \gg m)$

$m_\gamma^2 = -2E^2 + 2\vec{p}_1\vec{p}_2\cos\theta$

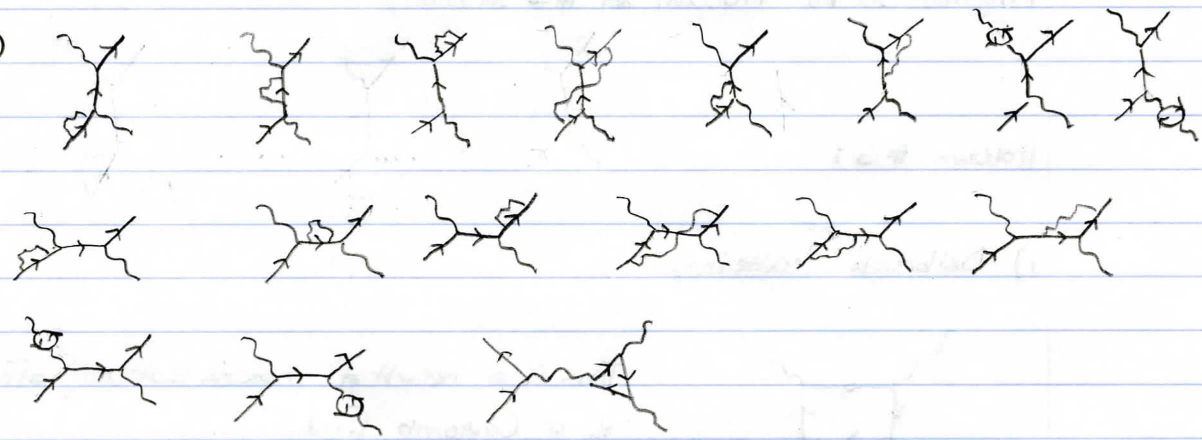
$m_\gamma^2 = -2E^2 - 2p_1p_2$ $\quad$ (elastic collision $\theta \sim 180°$)

$m_\gamma^2 = -4E^2$ $\quad (E \sim \vec{p})$

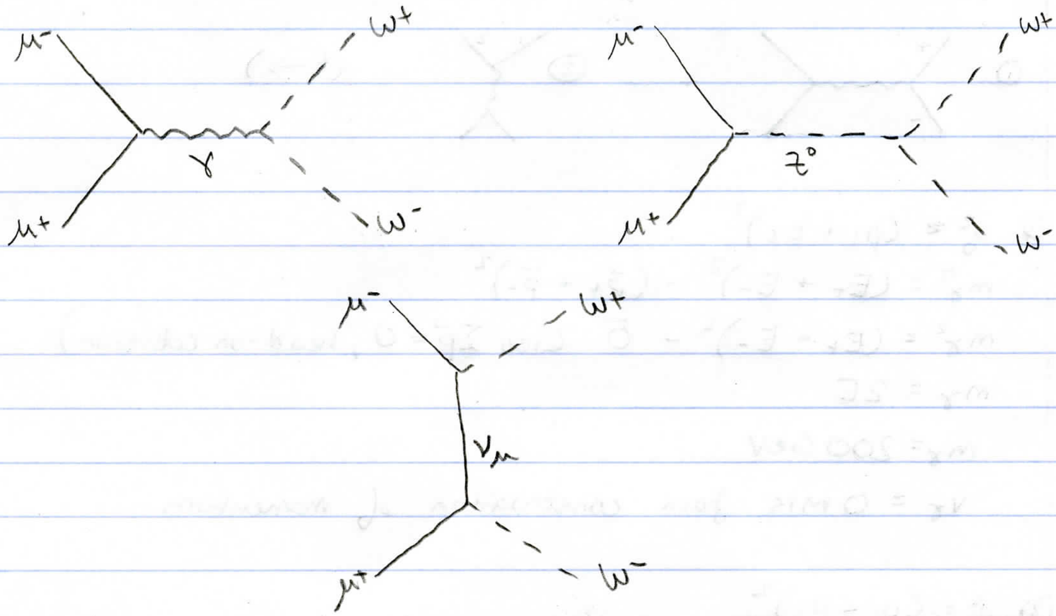
$v_\gamma = \vec{0}\,m/s$ again by conservation of momentum.

These values for m_γ and v_γ are not possible for real photons.

iii)



iv) There are three diagrams:

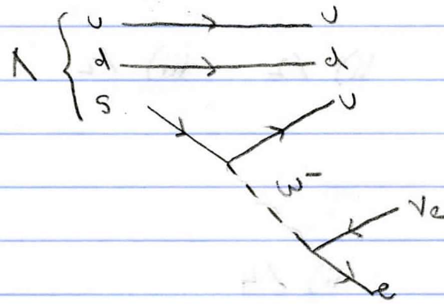


You need high energies because muons decay ($\tau_0 = 2.2 \times 10^{-6}$ s) ie. time dilation!

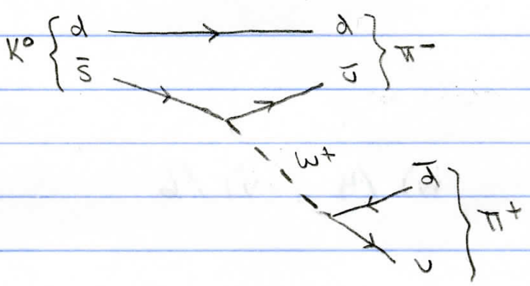
Physics 3575 - Problem Set #1 Solutions

v) $\pi^+ \to \mu^+ \nu_\mu$

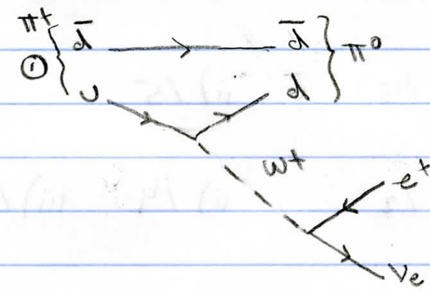
$\Lambda \to p e \bar{\nu}_e$



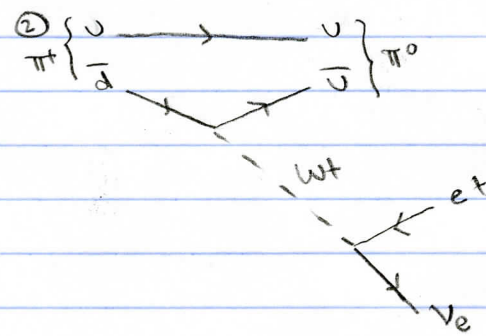
$K^0 \to \pi^+ \pi^-$



$\pi^+ \to \pi^0 e^+ \nu_e$



2) there are two diagrams here because π^0 quark content is $u\bar{u}$ or $d\bar{d}$.



You can tell these are weak decays because

① there are neutrinos

② you need a charged mediator (γ and gluons have 0e).

Mark Breakdown:

Q1 i) /5 ii) /2 iii) /2 iv) /1

Q2 /5

Q3 i) /1 ii) /4

Q4 i) /2 ii) /1 iii) /1 iv) /2 v) /3

Q5 i) /5 ii) /5

Q6 i) /2 ii) /4 iii) /5 iv) /4 v) /6

Total: /60