Problem Set #2 Solutions　　　　　　　　　　　　　　　　　　　　Physics 357S

1. $A + B \rightarrow C_1 + C_2 + \dots + C_n$ where A is the beam and B is the target.

a) $(p_\mu^A + p_\mu^B)^2 = (p_\mu^{c_1} + \dots + p_\mu^{c_n})^2$ by conservation of 4-momentum.

LHS in lab frame:
$$\begin{aligned}(p_\mu^A + p_\mu^B)^2 &= (\Sigma E)^2 - (\Sigma \vec{p})^2 \\ &= (E_A + E_B)^2 - (\vec{p}_A + \vec{0})^2 \quad (\text{since } B \text{ is target, } \vec{p}_B = 0) \\ &= E_A^2 + E_B^2 + 2E_A E_B - \vec{p}_A^{\,2} \\ &= E_A^2 - \vec{p}_A^{\,2} + E_B^2 + 2E_A E_B \\ &= m_A^2 + m_B^2 + 2E m_B\end{aligned}$$

RHS in COM frame:
$$\begin{aligned}(p_\mu^{c_1} + \dots + p_\mu^{c_n})^2 &= (\Sigma E)^2 - (\Sigma \vec{p})^2 \\ &= (E_{c_1} + E_{c_2} + \dots)^2 - (\vec{0} + \vec{0} + \dots)^2 \\ &= (E_{Tot})^2 \\ &= M^2\end{aligned}$$

(we are looking for threshold energy, so all C's have $\vec{p} = \vec{0}$)

Then, LHS = RHS

$$\begin{aligned}m_A^2 + m_B^2 + 2E m_B &= M^2 \\ 2E m_B &= M^2 - m_A^2 - m_B^2 \\ E &= (M^2 - m_A^2 - m_B^2)/2m_B\end{aligned}$$

→ just multiply this by c^2 if you aren't in natural units.

b) $e^+ + e^- \rightarrow \Upsilon(1S)$ where e^+ is the beam and e^- the target.

Then,
$$\begin{aligned}E &= (M_\Upsilon^2 - m_{e^+}^2 - m_{e^-}^2)/2m_{e^-} \\ &= (9.46\,\text{GeV}^2 - 2(511\,\text{keV})^2)/2(511\,\text{keV}) \\ &= 87.6\,\text{TeV}\end{aligned}$$

In the case of head-on collision:
$$\begin{aligned}(p_\mu^{e^+} + p_\mu^{e^-})^2 &= (p_\mu^\Upsilon)^2 \\ (E_{e^+} + E_{e^-})^2 - (\vec{p}_{e^+} + \vec{p}_{e^-})^2 &= m_\Upsilon^2 \\ (2E)^2 - (\vec{0})^2 &= m_\Upsilon^2 \\ (2E)^2 &= (m_\Upsilon)^2 \\ 2E &= m_\Upsilon \\ E &= (9.46\,\text{GeV})/2 \\ &= 4.73\,\text{GeV}\end{aligned}$$

(since $m_e \ll E$, $\vec{p} \sim E$)

c) $\pi^- + p \rightarrow \Lambda^0 + K^0$ where π^- is the beam and p is the target.

Then, using part A solution, $E_{\pi^-} = [(m_{\Lambda^0} + m_{K^0})^2 - m_{\pi^-}^2 - m_p^2] / 2m_p$

$$= [(1.116\ \text{GeV} + 497.6\ \text{MeV})^2 - (0.14\ \text{GeV})^2 - (0.981\ \text{GeV})^2] / 2(0.981\ \text{GeV})$$

$$\cong 0.83\ \text{GeV}$$

I was happy with $E \sim \vec{p}$, but in this case our pion is heavier than e^+, so to find the momentum

$$E^2 = \vec{p}^2 + m_\pi^2$$

$$\vec{p} = \sqrt{E^2 - m_\pi^2}$$

$$= \sqrt{(0.83\ \text{GeV})^2 - (0.14\ \text{GeV})^2}$$

$$\cong 0.82\ \text{GeV}$$

is better than simply stating $E \sim \vec{p} \cong 0.83$ GeV (though it is quite close!)

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Problem Set #2 Solutions Physics 3575

2. $A \rightarrow BC$ where A is at rest.

By conservation of 4-momentum,

$$p_\mu^A = p_\mu^B + p_\mu^C$$
$$(p_\mu^C)^2 = (p_\mu^A - p_\mu^B)^2$$
$$m_C^2 = (E_A - E_B)^2 - (\vec{p}_A - \vec{p}_B)^2$$
$$m_C^2 = E_A^2 - \vec{p}_A^{\,2} + E_B^2 - \vec{p}_B^{\,2} - 2E_AE_B + 2\vec{p}_A\vec{p}_B$$
$$m_C^2 = m_A^2 + m_B^2 - 2E_AE_B + \vec{0} \quad (\text{since } \vec{p}_A = \vec{0})$$
$$2E_AE_B = m_A^2 + m_B^2 - m_C^2$$
$$E_B = (m_A^2 + m_B^2 - m_C^2)/2m_A$$

For momentum, note that $|\vec{p}_B| = |\vec{p}_C|$ by conservation of momentum, so we only show the derivation for $\vec{p}_B$ since $\vec{p}_C$ follows.

Then, $E^2 = \vec{p}^{\,2} + m^2$

$$E_B = \vec{p}_B^{\,2} + m_B^2$$
$$\vec{p}_B^{\,2} = E_B^2 - m_B^2$$
$$\vec{p}_B^{\,2} = \frac{1}{4m_A^2}\left[m_A^2 + m_B^2 - m_C^2\right]^2 - m_B^2$$
$$\vec{p}_B^{\,2} = \frac{1}{4m_A^2}\left[m_A^4 + m_B^4 + m_C^4 - 2m_B^2m_C^2 - 2m_A^2m_C^2 + 2m_A^2m_B^2\right] - (m_B^2)\frac{4m_A^2}{4m_A^2}$$
$$\vec{p}_B^{\,2} = \frac{1}{4m_A^2}\left[m_A^4 + m_B^4 + m_C^4 - 2m_B^2m_C^2 - 2m_A^2m_C^2 - 2m_A^2m_B^2\right]$$
$$\vec{p}_B = \left(\frac{1}{2m_A}\right)\sqrt{\lambda(m_A^2, m_B^2, m_C^2)}$$

Now to find the COM energy for each decay product, use the result for E_B:

a) $E_{\mu^+} = 40.2$ GeV $\quad E_{\nu_\mu} = 40.2$ GeV

b) $E_{\gamma_1} = E_{\gamma_2} = 67.49$ MeV

c) $E_{\pi^+} = E_{\pi^-} = 2.64$ GeV

d) $E_\Lambda = 1.12$ GeV $\quad E_{\pi^0} = 0.19$ GeV

e) $E_\Lambda = 1.14$ GeV $\quad E_{K^-} = 0.54$ GeV

3. a) $K^+ + p \rightarrow K^+ + \pi^0 + p$ in LAB frame.

Final state: K^+ $\vec{p} = 5$ GeV $\left.\right\}$ $\theta = 7.3°$
π^0 $\vec{p} = 4$ GeV

The invariant mass: $M^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$
$= m_{K^+}^2 + m_{\pi^0}^2 + 2E_{K^+}E_{\pi^0} - 2\vec{p}_{K^+}\vec{p}_{\pi^0}$
$= m_{K^+}^2 + m_{\pi^0}^2 + 2E_{K^+}E_{\pi^0} - 2p_{K^+}p_{\pi^0}\cos\theta$

note: $E_{K^+}^2 = m_{K^+}^2 + \vec{p}_{K^+}^{\,2}$
$E_{K^+}^2 = (493.677 \text{ MeV})^2 + (5 \text{ GeV})^2$
$E_{K^+} = 5.024$ GeV $\sim \vec{p}_{K^+}$

Similarly, $E_{\pi^0} = 4.0024$ GeV $\sim \vec{p}_{\pi^0}$

Then, $M^2 = (493.677 \text{ MeV})^2 + (139.57 \text{ MeV})^2 + 2(5.024 \text{ GeV})(4.0024 \text{ GeV})$
$-2(5 \text{ GeV})(4 \text{ GeV})\cos(7.3°)$
$M^2 \simeq 0.8 \text{ GeV}^2$
$M \cong 0.896$ GeV
$M \cong 896$ MeV

From PDG, the K^* has a mass around ~ 896 MeV

b) Assuming $K^* \rightarrow K^+ + \pi^0$, the mean lifetime of K^* is calculated by,

K^* line width $\Gamma \sim 46$ MeV (pdg)
Then, $\tau = \hbar/\Gamma$
$= \hbar / 46 \text{ MeV}$
$\cong 1.4 \times 10^{-23}$ s

The lifetime is on the order of 10^{-23}s, which indicates that the strong force is resposible for this interaction. (see slide 8 of Lecture 9)

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c) Beam energy is $E_{K^+} = 20$ GeV (protons are our stationary target).

Before in LAB: $p^\mu_{K^+} = (E_{K^+}, \vec{p}_{K^+})$

$p^\mu_p = (m_p, \vec{0})$

After in COM: $p^\mu_{K^*} = (E_{K^*}, \vec{p}_{K^*})$

$p^\mu_p = (E_p, -\vec{p}_p)$ where $|\vec{p}_{K^*}| = |\vec{p}_p| = |$

Using conservation of invariant mass: $(p^\mu_{K^+} + p^\mu_p)^2 = (p^\mu_{K^*} + p^\mu_p)^2$

$$(E_{K^+} + m_p)^2 - (\vec{p}_{K^+})^2 = (E_{K^*} + E_p)^2 - (\vec{p} - \vec{p})^2$$

$$m_{K^+}^2 + m_p^2 + 2E_K m_p = E_{K^*}^2 + 2E_{K^*}E_p + E_p^2$$

$$m_{K^+}^2 + \cancel{m_p^2} + 2E_K m_p = m_{K^*}^2 + \vec{p}^2 + 2E_{K^*}E_p + \cancel{m_p^2} + \vec{p}^2$$

$$m_{K^+}^2 + 2E_K m_p = m_{K^*}^2 + 2\vec{p}^2 + 2\sqrt{m_{K^*} + \vec{p}^2}\sqrt{m_p^2 + \vec{p}^2}$$

$$(m_{K^+}^2 + 2E_K m_p - m_{K^*}^2 - 2\vec{p}^2)^2 = 4(m_{K^*} + \vec{p}^2)(m_p^2 + \vec{p}^2)$$

$$(m_K^2 + 2E_K m_p - m_{K^*}^2)^2 - 4\vec{p}^2(m_{K^+}^2 + 2E_K m_p - \cancel{m_{K^*}^2}) + \cancel{4\vec{p}^4}$$

$$= 4m_{K^*}^2 m_p^2 + 4\vec{p}^2(\cancel{m_{K^*}^2} + m_p^2) + \cancel{4p^4}$$

$$|\vec{p}| = \text{sqrt}\left[\frac{(m_{K^+}^2 + 2E_K m_p - m_{K^*}^2)^2 - 4m_{K^*}^2 m_p^2}{4(m_{K^+}^2 + 2E_K m_p + m_p^2)}\right]$$

units for mass/momentum are MeV $\longrightarrow$

$$= \text{sqrt}\left[\frac{(494^2 + 2(20000)(938) - 897^2)^2 - 4(897^2)(938^2)}{4(494^2 + 2(20000)(938) + 938^2)}\right]$$

$$\cong 2.97 \text{ GeV}$$

d) The invariant mass distribution of $K^+\pi^0$ will have a peak at the mass of the intermediate particle if that intermediate particle exists. If there is no intermediate particle, then there is no peak in the distribution at that mass. For this part, I expect graphs and axes to be labelled!

4. a) Lifetime and mean decay distance in LAB frame:

$$\tau_{LAB} = \gamma \tau_0$$
$$= \sqrt{1 + (p/m_0c^2)^2}\, \tau_0$$

from wikepedia

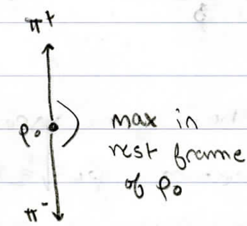
$$= \mathrm{sqrt}(1 + (5\,\mathrm{GeV}/769\,\mathrm{MeV})^2)(4.5\times10^{-24}\,\mathrm{s})$$
$$\cong 2.96\times10^{-23}\,\mathrm{s}$$

$$d_{LAB} = \beta c \tau_{LAB}$$
$$= (\sqrt{1 - 1/\gamma^2})\, c\, \tau_{LAB}$$
$$\cong 8.77\times10^{-15}\,\mathrm{m}$$

(Some students used the given decay width to find τ_0 and obtained slightly different answers - this was also fine.)

b) Theshold energy for $\pi^- + p \to \rho^0 + n$ can be found using the solution from #1, where $E_{TH} \doteq 1076$ MeV.

c) For the maximum opening angle, we start in the rest frame of ρ_0, where π^+ and π^- are back-to-back and $p^\mu_\rho = (m_\rho, \vec{0})$; $p^\mu_{\pi^+} = (E, \vec{p}) = p^\mu_{\pi^-}$.



$$p^\mu_\rho = p^\mu_+ + p^\mu_-$$
$$p^\mu_+ = p^\mu_\rho - p^\mu_-$$
$$(p^\mu_+)^2 = (p^\mu_\rho - p^\mu_-)^2$$
$$m_+^2 = m_\rho^2 + m_-^2 - 2p^\mu_\rho p^\mu_-$$
$$m_+^2 = m_\rho^2 + m_-^2 - 2m_\rho E_-$$
$$\Rightarrow E = (m_\rho^2 + m_+^2 - m_-^2)\left(\tfrac{1}{2m_\rho}\right)$$
$$= \left(\tfrac{1}{2}\right) m_\rho$$
$$= \left(\tfrac{1}{2}\right)(769\,\mathrm{MeV})$$
$$= 384.5\,\mathrm{MeV}$$

$$\therefore \vec{p} = \sqrt{E^2 - m_+^2}$$
$$= \sqrt{(384.5\,\mathrm{MeV})^2 - (139.57\,\mathrm{MeV})^2}$$
$$\doteq 358.27\,\mathrm{MeV}$$

Now we have to boost to the lab frame using Lorentz Transform:

$$\begin{bmatrix} E' \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ p_x \\ p_y \\ p_z \end{bmatrix} \Rightarrow \begin{aligned} \vec{p}_x' &= \gamma \vec{p}_x - \beta\gamma E \\ &= \beta\gamma E \end{aligned} \quad \text{and} \quad \vec{p}_y' = \vec{p}_y$$

Using $\vec{p}_p = 5\text{ GeV}$, $E_p = \sqrt{\vec{p}_p^{\,2} + m_p^2} \cong 5058.79$

$\gamma = 6.578$ (from a).

$v_p \cong \vec{p}_p / E_p \cong 0.9884c$

$\therefore \vec{p}_x' = \beta\gamma E$

$= (0.9884)(6.578)(384.5\text{ MeV})$

$\cong 2525\text{ MeV}$ (= 2500 MeV hopefully without rounding error!)
(↳ (see below).

$\vec{p}_y' \cong 358.27\text{ MeV}$

Then $\theta = 2\tan(\vec{p}_T/\vec{p}_L)$

$= 2\tan(\vec{p}_y/\vec{p}_x)$ (ie, $p_x = p_L$ = longitudinal; $p_y = p_T$ = transverse).

$= 2\tan(358.27/2500)$

$\cong 16.3°$

<u>note</u>: rather than going through this trouble, you can note that by conservation of momentum, $\vec{p}_L = \frac{1}{2} p_p = 2500$ MeV. Then you can see that $p_T' = p_T$, calculate the momentum in the COM, and find $\theta = 2\tan(358/2500) \cong 16.3$

The minimum angle is $\theta = 0°$, corresponding to the π^+ and π^- aligned with the direction of the parent.

5. a) $B = 8.7T$

From this, we can find the radius of the machine:

Centripetal force = Lorentz force

$$\gamma m v^2 (\tfrac{1}{r}) = qvB$$
$$\gamma m v (\tfrac{1}{r}) = qB$$
$$\Rightarrow r = \gamma m v / qB$$
$$r = P/qB \quad (\text{ie}, \ \vec{p} = \gamma m v)$$
$$= 6.5\,\text{TeV}/q\,8.7T$$
$$\cong 2.49\ \text{km}$$

b) Protons are not fundamental particles – they have contituents which carry some fraction of the proton's momentum.
The Higgs is produced along with other particles, so the collisions must have enough energy to produce all of these products.

c) SSC : $E = 20$ TeV per beam.

$T = 6T$

The centre of mass energy for the SSC is given by: $E_{CM} = 2E_{beam} = 40\,\text{TeV}$

An equivalent COM energy for a fixed target experiment would require a beam with an energy of: $E_{cm} = \sqrt{(E_p + m_p)^2 - (\vec{p})^2}$

$$= \sqrt{E_p^2 + m_p^2 + 2E_p m_p - \vec{p}^{\,2}}$$
$$= \sqrt{2m_p^2 + 2E_p m_p}$$
$$\cong \sqrt{2E_p m_p} \quad \text{since} \quad E \gg m$$
$$\Rightarrow E_p = E_{cm}^2 (\tfrac{1}{2m_p})$$
$$= (40\,\text{TeV})^2 / 2(938\ \text{MeV})$$
$$\cong 8.5 \times 10^5\ \text{TeV}$$

The diameter of the accelerator would be: $r = P/qB$

$$= (8.5 \times 10^5\ \text{TeV}) / q(6T)$$
$$\cong 4.74 \times 10^5\ \text{km}$$
$$\therefore d = 2r \cong 9.5 \times 10^5\ \text{km}$$

d) In this case, I believe the professor mistated the question. As long as a beam energy was quoted (ex// the energy calculated in part c) of $E_p = 8.5 \times 10^5$ TeV) then I was happy. :) (I believe some of you were asked to use a stationary electron target rather than protons).

In fact, I did mean electrons. That was to bring the effect of the tiny target mass on the CM energy. However, whether you sued electrons or protons is OK by me..... RSO

6. Linear collider with beam energy of 500 GeV. In this case, collide e^+ and e^-.

a) Assume heaviest particle possible is produced alone.

Then, $p_*^{\mu\,2} = (p_+^\mu + p_-^\mu)^2$ where p_*^μ = 4 momentum of new particle

$$m_*^2 = (E_+ + E_-)^2 - (\vec{p}_+ + \vec{p}_-)^2$$
$$= m_+^2 + m_-^2 + 2E_+E_- - 2\vec{p}_+\vec{p}_- \quad (E >> m)$$
$$= 4E^2 \text{ (since } E_+ = E_-)$$
$$\Rightarrow m_* = 2E$$
$$= 2(500\text{ GeV})$$
$$= 1\text{ TeV}$$

The momentum of m_* in the LAB is $\vec{p}_* = \vec{0}$, since we consider a head on collision and take m_* to be the maximum possible mass (ie, it is produced at threshold).

b) The structure size we could resolve is: $\lambda = h/p$ (de Broglie)

$$= h/500\text{ GeV}$$
$$= 2.48 \times 10^{-18}\text{ m}$$

For SLAC, $\lambda = h/p = h/40\text{ GeV} = 3.09 \times 10^{-17}\text{ m}$

The present limits on the size of the quarks are $\sim 10^{-19}$ m. I saw different #s quoted, so whether you could resolve quarks depended on that.

c) For $\vec{p} = 500$ GeV, the velocity of the electron is given by $v = p/\gamma m$

$$= (500\text{ GeV})/\gamma(511\text{ keV})$$
$$\simeq 0.999999981\,c$$

note: $\gamma = \sqrt{1 + (p/m_0c^2)^2}$

$$= \sqrt{1 + (500\text{ GeV}/511\text{ keV})^2}$$
$$= 978473.6$$

$\therefore$ the electrons travel very close to the speed of light.

(aside: many students calculated the speed of electrons from SLAC, I also considered this acceptable).

d) Recall that $\vec{F} = q\vec{E}$ and $E = Fd$ (ie, work).

Then, $E = q\vec{E}d$

$$\Rightarrow \vec{E} = E/qd$$
$$= (500\text{ GeV})/q(15\text{ km})$$
$$= 3.3\times10^{7}\text{ V/m}$$

$\therefore$ the effective accelerating field is $\vec{E} = 3.3\times10^{7}\text{ V/m}$. (for 40 GeV, $\vec{E} \cong 2.7\times10^{6}\text{ V/m}$)

e) Newtonian mechanics says $E = \frac{1}{2}mv^2$. Then, $\frac{1}{2}mv^2 = q\vec{E}d$

$$d = \frac{1}{2}mv^2\left(\frac{1}{q\vec{E}}\right)$$
$$= \frac{1}{2}(511\text{ keV})(c)^2\left(\frac{1}{e\cdot3.3\times10^{7}\text{ V/m}}\right)$$
$$\cong 0.0077\text{ m}$$

$\therefore$ if newtonian mechanics applied the electrons would have to travel ~7.7 mm.

f) The accelerator moves at a speed v in the electron's frame.

Also note that we need to take into account length contraction.

$\therefore$ $d' = d/\gamma$ where $d = 15$ km as in part d)

$$= 15\text{ km}/\left(\sqrt{1-v^2/c^2}\right)^{-1}$$
$$= 15\text{ km}\cdot\left(1/978473\right) \text{ using } \gamma \text{ from c)}$$
$$\cong 0.0153\text{ m}$$

Marking Scheme:

#1. a) /3
b) /3
c) /2

#2. /4
a-e) /8

#3. a) /4
b) /2
c) /4
d) /2

#4. a) /2
b) /2
c) /4

#5. a) /2
b) /2
c) /4
d) /2

#6. a) /4
b) /4
c) /2
d) /4
e) /2
f) /2

Total: /68