PHY 357 Problem Set #1 Solutions January 2016

Question #1

This question discusses two different Lorentz frames: the frame of the kaon moving close to the speed of light; and the frame of the experimental hall, which also happens to be the frame of the second kaon since it is produced at rest. The question describes how, depending on the frame, the decays of the kaons occur at different orders in time.

a) The time ordering is not a problem as long as the two kaons are not causally connected. In special relativity, events which are space-like separated can occur at different time-orders in different frames.

b) So, in order for this not to be a problem, we have to explicitly show that Ks and Ks' are space-like. I will work in the experimental hall frame. We are looking for Δs^2.

In the hall frame, the lifetime of the kaon moving at speed v is $\left(\tau/\sqrt{1-v^2/c^2}\right)$
the lifetime of the kaon at rest is (τ)

From this we can write the $c^2\Delta t^2$ term from $\Delta s^2 = \Delta x^2 - c^2\Delta t^2$ in the experimental hall frame:

$$c^2\Delta t^2 = c^2\left[\tau - \tau/\sqrt{1-v^2/c^2}\right]^2$$

Now we need to know Δx in the experimental hall frame as well. From the question, we know that the kaon at speed v decays in the second detector, while the kaon at rest decays in the first detector, which is where both kaons originated. Therefore Δx will simply be the distance traveled by the kaon moving at speed v in the experimental hall frame:

$$\Delta x^2 = (vt)^2 = \left[v\left(\tau/\sqrt{1-v^2/c^2}\right)\right]^2$$

Now we need to show that these events are spacelike ie. $\Delta s^2 > 0$

$$\begin{aligned}
\Delta s^2 &= \Delta x^2 - c^2\Delta t^2 \\
&= \left[v\left(\tau/\sqrt{1-v^2/c^2}\right)\right]^2 - \left[c\left(\tau - \tau/\sqrt{1-v^2/c^2}\right)\right]^2 \\
&= v^2\tau^2/(1-v^2/c^2) - c^2\tau^2\left[1 - 1/\sqrt{1-v^2/c^2}\right]\left[1 - 1/\sqrt{1-v^2/c^2}\right] \\
&= v^2\tau^2/(1-v^2/c^2) - c^2\tau^2 + 2c^2\tau^2/\sqrt{1-v^2/c^2} - c^2\tau^2/(1-v^2/c^2) \\
&= \frac{v^2\tau^2 - c^2\tau^2}{1-v^2/c^2} + \frac{2c^2\tau^2}{\sqrt{1-v^2/c^2}} - c^2\tau^2
\end{aligned}$$

Question #1b) continued...

$$\Delta s^2 = \frac{c^2\tau^2(v^2/c^2 - 1)}{1 - v^2/c^2} + \frac{2c^2\tau^2}{\sqrt{1-v^2/c^2}} - c^2\tau^2$$

$$= \frac{-c^2\tau^2(1-v^2/c^2)}{1-v^2/c^2} + \frac{2c^2\tau^2}{\sqrt{1-v^2/c^2}} - c^2\tau^2$$

$$= -c^2\tau^2 - c^2\tau^2 + 2c^2\tau^2/\sqrt{1-v^2/c^2}$$

$$= 2c^2\tau^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

We know that $\frac{1}{\sqrt{1-v^2/c^2}} > 1$, therefore Δs^2 must be positive and Ks and Ks' are space-like separated.

Question #2:

In this question, we are concerned with two decays: $\pi^+ \to \mu^+ \nu_\mu$

$K^+ \to \mu^+ \nu_\mu$

The calculations for each decay are very similar, I will therefore only work out the π^+ calculation. All of our "number of ..." units will be in per seconds.

For the π^+: $m_\pi = 0.14\ \text{GeV}/c^2$ For the beam: $d = 800\,\text{m}$

$\tau_0 = 2.6\times10^{-8}\,\text{s}$ $\vec{p} = 400\ \text{GeV}/c$

$N_0 = 10^{15}$ per second

We work with the decay equation: $N_{\pi^+} = N_0^{\pi^+} e^{-t/\tau_{\pi^+}}$

To find the number of neutrinos per second at the end of the tunnel, we need to know the mean lifetime in the lab frame and the time t that it takes for the beam to travel along the tunnel.

Mean lifetime: $\tau = \gamma\tau_0$

Travel time: $t = \gamma m d/\vec{p}$ ie. $\vec{p} = \gamma m v = \gamma m (d/t)$

Question #2 continued ...

$$N_{\pi^+}' = N_0^{\pi^+} \exp\left[-\frac{\gamma m d}{p} \cdot \frac{1}{\gamma \tau_{\pi^+}}\right]$$
$$= (10^{15}) \exp\left[-(0.14\,GeV/c^2)(800\,m)/(400\,GeV/c)(2.6\times10^{-8}s)\right]$$
$$= 9.6\times10^{14}$$ pions decay per second, $\therefore N_{\pi^+} = 10^{15} - 9.6\times10^{14} = 3.5\times10^{13}$ neutrinos per second.

Similarly for K^+, we get $N_{K^+} = 2.4\times10^{14}$ neutrinos produced from Kaons per second.

Therefore the number of neutrinos at the end of the tunnel per second is.

$$N_\nu = N_{\pi^+} + N_{K^+}$$
$$= 2.75\times10^{14} \text{ neutrinos/second.}$$

Question #3:

For this question, we have to place a limit on the neutrino mass based on the measurement of incoming neutrinos from a supernova SN1987A from 1987. Kamioka and IMB observed neutrinos from SN1987A with the following properties:

$\Delta t = 10s$

$E = 10-40$ MeV .

So, over a period of 10s, neutrinos within an energy range of 10-40 MeV reached Earth. The key to this question is to realize that if two particles are produced at the same time, and travel to the same distant location; the more energetic particles will arrive earlier if these particles have mass,

We assume that the supernova produced all of the neutrinos at once.

Then, @ $t = 0s$ $E_1 = 40$ MeV

$t = 10s$ $E_2 = 10$ MeV

Taking the ratio of
$$E_1/E_2 = \gamma_1 mc^2/\gamma_2 mc^2$$
$$E_1/E_2 = \gamma_1/\gamma_2$$
$$40/10 = \gamma_1/\gamma_2$$
$\gamma_1/\gamma_2 = 4 \Rightarrow$ This gives γ_1 in terms of γ_2 : $\gamma_1 = 4\gamma_2$

Question #3 continued ...

We can also write out expressions for the time the neutrinos traveled from SN1987A to Earth:

$t_1 = d/v_1$ $\Rightarrow$ Since $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, we can write $v = c\sqrt{1-\frac{1}{\gamma^2}}$
$t_2 = d/v_2$

Therefore we can write $t_1 = d\left[c\sqrt{1-\frac{1}{\gamma_1^2}}\right]^{-1}$

$= (d/c)\left(1/\sqrt{1-1/\gamma_1^2}\right)$ note: d/c = time for light to travel = t_0

$= t_0 \, 1/\sqrt{1-1/\gamma_1^2}$

$\cong t_0 + \frac{1}{2} t_0 \frac{1}{\gamma_1^2}$

Similarly, we can write $t_2 \cong t_0 + \frac{1}{2} t_0 \frac{1}{\gamma_2^2}$

Then, $t_2 - t_1 = \frac{1}{2} t_0 \left(\frac{1}{\gamma_2^2} - \frac{1}{\gamma_1^2}\right)$

$t_2 - t_1 = \frac{1}{2} t_0 \left(\frac{1}{\gamma_2^2}\left(1 - 1/4^2\right)\right)$ (remember, $\gamma_1 = 4\gamma_2$!)

$10\,s = 0.46875\, t_0/\gamma_2^2$

$\Rightarrow \gamma_2 = \sqrt{0.046875\,s^{-1}\, t_0}$

Now we can use the energy E_2 to find a limit on the neutrino mass:

$E_2 = \gamma_2 m c^2$

$E_2 = \sqrt{0.046875\,s^{-1}\,t_0}\, mc^2$

$m = E_2 / \sqrt{0.046875\,s^{-1}\,t_0}\, c^2$

$= (10\text{ MeV}) / \sqrt{(0.046875\,s^{-1})(1.5\times10^5\,y)}$

$= 21.2\text{ eV}$

$\therefore$ our limit is $m_\nu \lesssim 21.2\text{ eV}$

Question #4

a) We know that these are Lorentz-invariant because these are combinations of Lorentz invariant quantities.

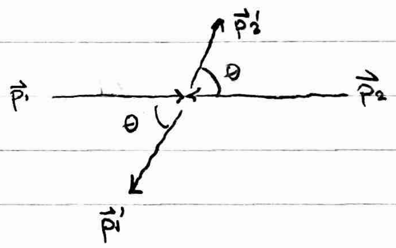
$$s \equiv (p_A + p_B)^2$$
$$= (p_A + p_B)(p_A + p_B)$$
$$= \underbrace{p_A^2}_{\text{invariant}} + \underbrace{p_B^2}_{\text{invariant}} + \underbrace{2p_A p_B}_{\text{invariant}}$$

You can check explicitly that the scalar product of two different 4-vectors is invariant by simply taking the Lorentz transform. t and u follow similarly.

b) Now we are working in the centre of mass frame in the special case of elastic 2body scattering. The particles have the same mass m.

Before the collision $\vec{p}_1 = -\vec{p}_2$

After the collision $\vec{p}_1{}' = \vec{p}_2{}'$



Then we can write:

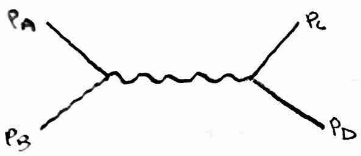
$$s \equiv (p_A + p_B)^2$$
$$= (p_1 + p_2)^2$$
$$= p_1^2 + p_2^2 + 2p_1 p_2$$
$$= 2m^2 + 2(E_1E_2 - |\vec{p}_1||\vec{p}_2|)$$
$$= 2m^2 + 2E^2 + 2|\vec{p}|^2 \cos(180°)$$
$$= 2m^2 + 2E^2 + 2|\vec{p}|^2 \quad \underline{\text{note}}: E^2 = p^2 + m^2$$
$$= 2m^2 + 2(|\vec{p}|^2 + m^2) + 2|\vec{p}|^2$$
$$= 4(m^2 + |\vec{p}|^2)$$

$$t \equiv (p_A - p_C)^2$$
$$= (p_1 - p_1')^2$$
$$= (E_1 - E_1')^2 - (\vec{p}_1 - \vec{p}_1{}')^2$$
$$= 0 - \vec{p}_1{}^2 - \vec{p}_1{}'^2 + 2\vec{p}_1\vec{p}_1{}'$$
$$= -2|\vec{p}|^2 + 2|\vec{p}|^2\cos\theta$$
$$= -2|\vec{p}|^2(1 - \cos\theta)$$

$$u \equiv (p_A - p_D)^2$$
$$= (p_1 - p_2')^2$$
$$= (E_1 - E_2')^2 - (\vec{p}_1 - \vec{p}_2{}')^2$$
$$= -\vec{p}_1{}^2 - \vec{p}_2{}'^2 + 2\vec{p}_1\vec{p}_2{}'$$
$$= -2|\vec{p}|^2 + 2|\vec{p}|^2\cos(180 - \theta)$$
$$= -2|\vec{p}|^2(1 + \cos\theta)$$

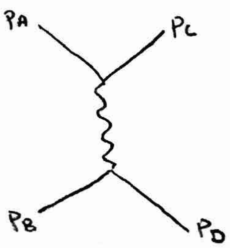
Question #4 continued...

c) For the timelike process:



$$q^2 = (p_A + p_B)^2$$
$$= m_A^2 + m_B^2 + 2p_A p_B$$
$$= m_A^2 + m_B^2 + 2(E_A E_B - \vec{p}_A \vec{p}_B)$$
$$= m_A^2 + m_B^2 + 2E_A E_B - 2|\vec{p}_A||\vec{p}_B|\cos\theta \quad (\text{in COM } \cos\theta = -1)$$
$$= m_A^2 + m_B^2 + 2E_A E_B + 2|\vec{p}_A||\vec{p}_B|$$
$$\therefore q^2 > 0$$

For the spacelike process:



$$q^2 = (p_A - p_C)^2$$
$$= m_A^2 + m_C^2 - 2(E_A E_C - \vec{p}_A \vec{p}_C)$$
$$= m_A^2 + m_C^2 - 2E_A E_C + 2|\vec{p}_A||\vec{p}_C|\cos\theta$$
$$\Rightarrow 2E_A E_C > m_A^2 + m_C^2$$
$$\Rightarrow 2|\vec{p}_A||\vec{p}_C|\cos\theta < 2E_A E_C$$
$$\therefore q^2 < 0$$

Question #5

N_3 depends on decay of N_2 and on decay of N_3 $\therefore$ $dN_3/dt = \lambda_2 N_2 - \lambda_3 N_3$. If N_3 is stable, then $dN_3/dt = \lambda_2 N_2$.

For the integrals,

$\boxed{N_1}$ $$\frac{dN_1}{dt} = -\lambda_1 N_1 \Rightarrow \int \frac{dN_1}{N_1} = -\int \lambda_1 dt$$

$$\ln(N_1/N_1^0) = -\lambda_1 t$$
$$N_1 = N_1^0 e^{-\lambda_1 t}$$

Question #5 continued...

$\boxed{N_2}$ $\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$

$e^{\lambda_2 t}\frac{dN_2}{dt} = \lambda_1 N_1^o e^{-\lambda_1 t} e^{\lambda_2 t} - \lambda_2 N_2 e^{\lambda_2 t}$

$e^{\lambda_2 t}\frac{dN_2}{dt} + \lambda_2 N_2 e^{\lambda_2 t} = \lambda_1 N_1^o e^{-\lambda_1 t} e^{\lambda_2 t}$

$\frac{d}{dt}\left(N_2 e^{\lambda_2 t}\right) = \lambda_1 N_1^o e^{-\lambda_1 t} e^{\lambda_2 t}$

$\int d(N_2 e^{\lambda_2 t}) = \int \lambda_1 N_1^o e^{-\lambda_1 t} e^{\lambda_2 t}\, dt$

$N_2 e^{\lambda_2 t} = \lambda_1 (\lambda_2 - \lambda_1)^{-1} N_1^o e^{-\lambda_1 t} e^{\lambda_2 t} + C$

To find C: $N_2(0) = 0$ so $N_2(0) e^{\lambda_2(0)} = \lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o e^{-\lambda_1(0)} e^{\lambda_2(0)} + C$

$0 = \lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o + C$

$\Rightarrow C = -\left(\frac{\lambda_1}{\lambda_2-\lambda_1}\right) N_1^o$

$\therefore N_2 e^{\lambda_2 t} = \lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o e^{-\lambda_1 t} e^{\lambda_2 t} - \lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o$

$N_2 = \frac{\lambda_1}{\lambda_2-\lambda_1} N_1^o \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$

$\boxed{N_3}$ $\frac{dN_3}{dt} = \lambda_2 N_2$

$dN_3 = \lambda_2 N_2\, dt$

$dN_3 = \lambda_2\lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) dt$

$N_3 = \lambda_2\lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o \int e^{-\lambda_1 t}\, dt - \int e^{-\lambda_2 t}\, dt$

$N_3 = \lambda_2\lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o \left[e^{-\lambda_1 t}\left(-\frac{1}{\lambda_1}\right) - e^{-\lambda_2 t}\left(-\frac{1}{\lambda_2}\right)\right] + C$

$N_3 = \lambda_2\lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o \left[e^{-\lambda_2 t}/\lambda_2 - e^{-\lambda_1 t}/\lambda_1\right] + C$

For C: $N_3(0) = 0 = \lambda_2\lambda_1(\lambda_2-\lambda_1)^{-1} N_1^o \left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right] + C$

$C = -\lambda_2\lambda_1(\lambda_2-\lambda_1) N_1^o \left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right]$

$\therefore N_3 = \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} N_1^o \left[\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{1}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1} + \frac{1}{\lambda_1}\right]$

$= \frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1} N_1^o \left[\frac{1}{\lambda_1}\left(1-e^{-\lambda_1 t}\right) - \frac{1}{\lambda_2}\left(1-e^{-\lambda_2 t}\right)\right]$

Question #5 continued ...

(a) $\lambda_1 > \lambda_2$ "no-equilibrium" - the half-life of the daughter is larger than that of the parent, so as $t \gg \lambda_1^{-1}$,

$$N_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1^0 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$
$$N_2 \sim \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1^0 \left(0 - e^{-\lambda_2 t}\right)$$
$$N_2 \sim e^{-\lambda_2 t}$$

→ the daughter's activity grows according to the equation for N_2, until, after enough time has passed, <u>only the daughter's activity is left</u>.

(b) $\lambda_1 < \lambda_2$ "transient equilibrium" - the half-life of the parent is larger than that of the daughter, so as $t \gg \lambda_2^{-1}$,

$$N_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1^0 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$
$$N_2 \sim \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1^0 e^{-\lambda_1 t}$$
$$N_2 \sim \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1$$
$$\frac{N_2}{N_1} \sim \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

→ like before, at first the daughter grows according to the N_2 equation, but after enough time, N_1 and N_2 are present in a constant ratio and all activities decay with the half-life of the parent.

b) $^{210}_{83}Bi \xrightarrow{\beta} 7.2 \text{ days} \rightarrow {}^{210}_{84}Po \xrightarrow{\alpha} 200 \text{ days} \rightarrow {}^{206}_{82}Pb$

At $t=0$, $N_{Bi} = 100\%$, $N_{Po} = 0$, $N_{Pb} = 0$

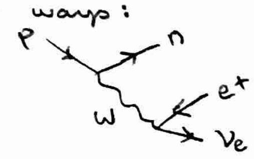
For the time at which α-emission is at a maximum, we need to find t at $dN_2/dt = 0$

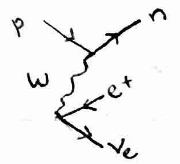
$$\frac{dN_2}{dt} = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1^0 \frac{d}{dt}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$
$$0 = -\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}$$
$$\Rightarrow \lambda_1 e^{-\lambda_1 t} = \lambda_2 e^{-\lambda_2 t}$$
$$\frac{\lambda_1}{\lambda_2} = e^{-\lambda_2 t} / e^{-\lambda_1 t}$$
$$\ln(\lambda_1/\lambda_2) = (\lambda_1 - \lambda_2)\, t$$
$$\ln(200/7.2) = \left(\frac{1}{7.2} - \frac{1}{200}\right) t$$
$$\Rightarrow t \cong 24.8 \text{ days}$$

Question #6

Virtual particles do not have arrows because it is meaningless to talk about their direction. For example, we can consider a β-decay diagram with a W propagator. We can draw the diagram in a few ways:

ex#1

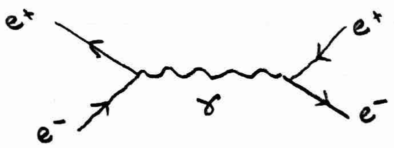


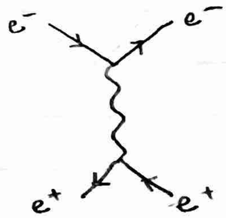


→ so we can look it as a W^+ from the quark vertex to the positron/neutrino vertex OR as a W^- goes from the positron/neutrino to the quarks

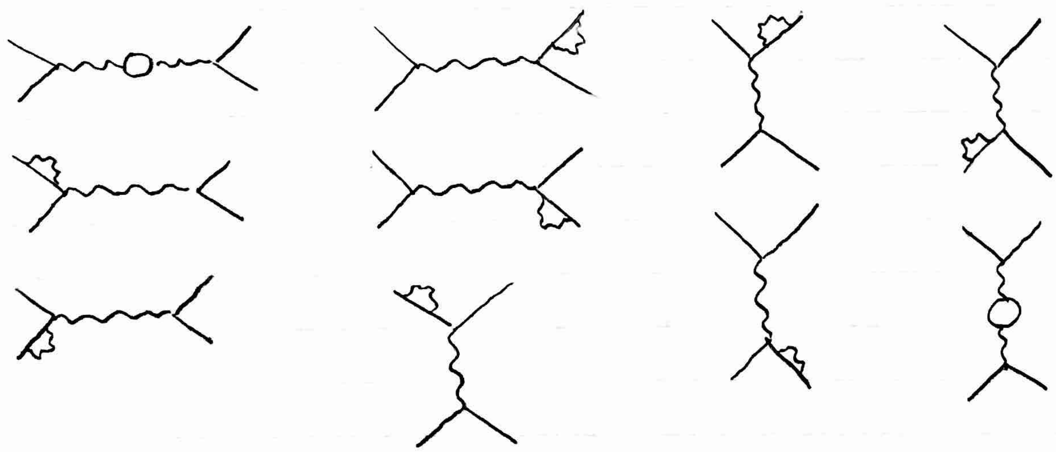
a) $e^+ + e^- \longrightarrow e^+ + e^-$ Feynman diagrams:

1st Order:



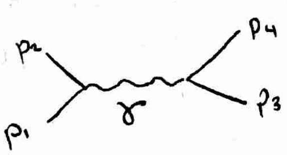


2nd Order (just 10 - but there were more options!)



b) Bhabha scattering for electron/positron at rest: there are two diagrams to consider:

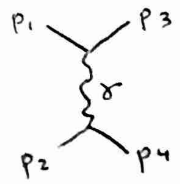
①



$$p_\gamma^2 = (p_1 + p_2)^2$$
$$m_\gamma^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$
$$m_\gamma^2 = (E_1 + E_2)^2$$
$$m_\gamma^2 = (2m_e)^2$$
$$m_\gamma = 2m_e$$

Since $\vec{p}_1$ and $\vec{p}_2$ are at rest, by conservation of momentum the photon must have $\vec{v}_\gamma = \vec{0}$ m/s

Question #6 continued . . .

② 

$p_\gamma^2 = (p_1 - p_3)^2$
$m_\gamma^2 = (E_1 - E_3)^2 - (\vec{p}_1 - \vec{p}_3)^2$
$m_\gamma^2 = 0$

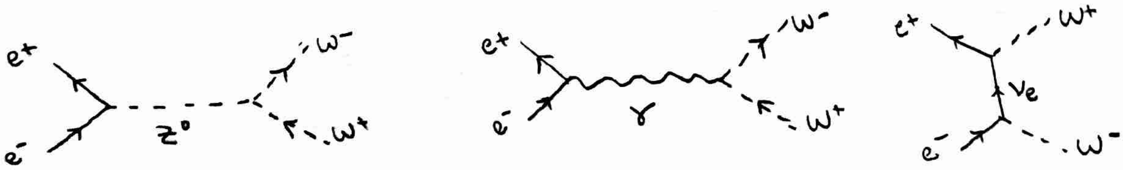
Again, by conservation of momentum, $\vec{v}_\gamma = \vec{0}$ m/s

Neither situation is possible for real photons.

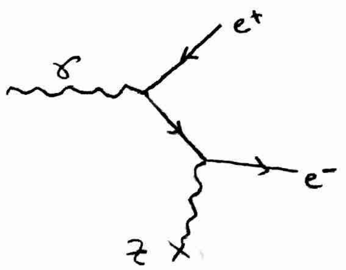
c) Now for the same diagrams, except e^+ and e^- have finite momentum:

① Annihilation
$q^2 = (p_1 + p_2)^2$
$q^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$
$q^2 = (E_1 + E_2)^2$ (ie., $\Sigma \vec{p} = \vec{0}$ in COM frame)
$q^2 = (2E)^2$
$q = 2E \Rightarrow 2E > 0 \therefore q^2$ is positive $\therefore$ timelike

② Elastic Scattering
$q^2 = (p_1 - p_3)^2$
$= p_1^2 + p_3^2 - 2p_1p_3$
$= m_1^2 + m_2^2 - 2(E_1E_3 - \vec{p}_1\vec{p}_3)$
$= m_1^2 + m_2^2 - 2E_1E_3 + 2\vec{p}_1\vec{p}_3 \cos\theta$
$= m_1^2 + m_2^2 - 2E_1E_3 - 2|\vec{p}_1||\vec{p}_3|$ (ie., elastic, $\theta = 180°$)
$\Rightarrow 2E_1E_3 - 2|\vec{p}_1||\vec{p}_3| > m_1^2 + m_2^2 \therefore q^2$ is negative $\therefore$ spacelike

d) 

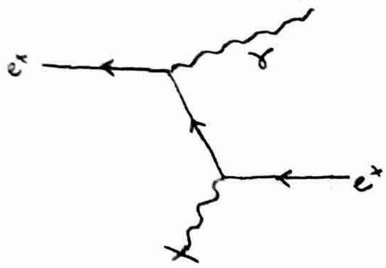
e) Pair Production $\gamma \to e^+e^-$



$p_\gamma^2 = (p_+ + p_-)^2$ in rest frame of photon.
$0 = (E_+ + E_-)^2 - (\vec{p}_+ + \vec{p}_-)^2$
$0 = (E_+ + E_-)^2 \Rightarrow$ impossible

Question #6 continued ...

Bremsstrahlung $e^+ \rightarrow \gamma e^+$



$p_e^2 = (p_e + p_\gamma)^2$ in rest frame of photon

$m_e^2 = m_e^2 + 0 + 2(E_e E_\gamma - \vec{p_e}\vec{p_\gamma})$

$m_e^2 = m_e^2 + 2E_e E_\gamma \Rightarrow$ impossible.

From the 4-vectors, we can see that, in the absense of the field of a nucleus, both pair production and Bremsstrahlung do not conserve 4-momentum.