PHY 357 Problem Set #2 Solutions February 2016

Question #1

$A + B \rightarrow C_1 + C_2 + \ldots + C_n$ @ threshold, where A is the beam and B is the target.

a) $(p_\mu^A + p_\mu^B)^2 = (p_\mu^{C_1} + p_\mu^{C_2} + \ldots + p_\mu^{C_n})^2$ by conservation of 4-momentum

LHS in LAB frame: $(p_\mu^A + p_\mu^B)^2 = (E_A + E_B)^2 - (\vec{p}_A + \vec{p}_B)^2$

$= (E_A + E_B)^2 - (\vec{p}_A + \vec{0})^2$ (since B is target, $\vec{p}_B = \vec{0}$)

$= E_A^2 - \vec{p}_A^{\,2} + E_B^2 + 2E_A E_B$

$= m_A^2 + m_B^2 + 2E_A m_B$

RHS in COM frame: $(p_\mu^{C_1} + p_\mu^{C_2} + \ldots + p_\mu^{C_n})^2 = (\Sigma E_i)^2 - (\Sigma \vec{p}_i)^2$

$= (\Sigma E_i)^2 - (\Sigma \vec{0})^2$ (we are looking for threshold energy, so all C's have $\vec{p} = 0$!)

$= (\Sigma m)^2$

$= M^2$ where $M = m_1 + m_2 + \ldots + m_n$

Then LHS = RHS: $m_A^2 + m_B^2 + 2E_A m_B = M^2$

$$\Rightarrow E_A = \frac{(M^2 - m_A^2 - m_B^2)}{2m_B}$$

b) $e^+ + e^- \longrightarrow \Psi(c\bar{c})$ for (1) fixed-target

(2) collider

(1) For fixed target, $\vec{p}_- = \vec{0}$ so

$$E_+ = \frac{(m_\Psi^2 - m_+^2 - m_-^2)}{2m_-}$$

$$= \frac{(3.1\,\text{GeV}^2 - 2(511\,\text{keV})^2)}{2(511\,\text{keV})}$$

$= 9400$ GeV

(2) For collider, with $\vec{p}_+ = \vec{p}_-$, $(p_+ + p_-)^2 = (p_\Psi)^2$

$(E_+ + E_-)^2 - (\vec{p}_+ + \vec{p}_-)^2 = (p_\Psi)^2$

$(2E)^2 - (\vec{0}) = m_\Psi^2$

$2E = m_\Psi$

$E = 3.1\,\text{GeV}/2$

$= 1.55$ GeV

Question #1 continued ...

c) $\pi^- + p \rightarrow \Delta^{++} + \pi^- + \pi^-$, pion beam and LH target

$$E_\pi = \frac{(m_\Delta + m_\pi + m_\pi)^2 - m_\pi^2 - m_p^2}{2m_p}$$

$$= \frac{(1.323\,GeV + 2(0.14\,GeV))^2 - 0.14\,GeV^2 - 0.981\,GeV^2}{2(0.981\,GeV)}$$

$$\cong 0.81\,GeV \quad \therefore \vec{p} = \sqrt{E^2 - m^2} \cong 0.798\,GeV \cong 0.8\,GeV$$

Question #2

$A \rightarrow B + C$ where A is at rest

By conservation of 4-momentum, $p_A = p_B + p_C$

$$p_A - p_B = p_C$$

$$(p_A - p_B)^2 = p_C^2$$

$$m_A^2 + m_B^2 - 2(E_A E_B - 2\vec{p}_A\vec{p}_B) = m_C^2$$

$$m_A^2 + m_B^2 - 2m_A E_B = m_C^2 \quad (\text{since } \vec{p}_A = \vec{0})$$

$$\Rightarrow E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A}$$

For the momentum, we also will need $E_C = {(m_A^2 + m_C^2 - m_B^2)}/{2m_A}$, which you can find similarly to E_B.

Then, to find $|\vec{p}_C| = |\vec{p}_B| = |\vec{p}|$, we use: $p_A^2 = (p_B + p_C)^2$

$$m_A^2 = m_B^2 + m_C^2 + \underbrace{2E_BE_C}_{*} - 2|\vec{p}|^2$$

$$* = 2\left[\frac{m_A^2 + m_B^2 - m_C^2}{2m_A}\right]\left[\frac{m_A^2 + m_C^2 - m_B^2}{2m_A}\right]$$

$$= (2/4m_A^2)\left[m_A^4 + \cancel{m_A^2 m_C^2} - \cancel{m_A^2 m_B^2} + \cancel{m_B^2 m_A^2} + m_B^2 m_C^2 - m_B^4 - \cancel{m_C^2 m_A^2} - m_C^4 + m_C^2 m_B^2\right]$$

$$= (1/2m_A^2)\left[m_A^4 - m_B^4 - m_C^4 + 2m_B^2 m_C^2\right]$$

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Question #2 continued ...

Rearranging ⊛ and subbing * back in:

$m_A^2 = m_B^2 + m_C^2 + 2E_BE_C - 2|\vec{p}|^2$

$\Rightarrow |p|^2 = \frac{1}{2}(2E_BE_C - m_A^2 + m_B^2 + m_C^2)$

$= (\frac{1}{2})(\frac{1}{2m_A^2})[m_A^4 - m_B^4 - m_C^4 + 2m_B^2m_C^2 - (2m_A^2)(m_A^2 - m_B^2 - m_C^2)]$

$= \frac{1}{4m_A^2}[m_A^4 - m_B^4 - m_C^4 + 2m_B^2m_C^2 - 2m_A^4 + 2m_A^2m_B^2 + 2m_A^2m_C^2]$

$= \frac{1}{4m_A^2}[-m_A^4 - m_B^4 - m_C^4 + 2m_B^2m_C^2 + 2m_A^2m_B^2 + 2m_A^2m_C^2]$

$\therefore |\vec{p}| = \frac{\sqrt{\lambda(m_A^2, m_B^2, m_C^2)}}{2m_A^2}$

a) E_{cm} for $K^+ \longrightarrow \mu^+ + \nu_\mu \Rightarrow E_{\mu^+} = \frac{(493.65\text{ MeV})^2 + (105.7\text{ MeV})^2 - 0}{2(493.65\text{ MeV})}$

$\cong 258.1\text{MeV}$

$\Rightarrow E_{\nu_\mu} = E_{K^+} - E_{\mu^+}$

$= (493.65\text{ MeV}) - (258.1\text{ MeV})$

$\cong 235.5\text{ MeV}$

Similarly, b) $\eta^0 \rightarrow \gamma\gamma \Rightarrow E_\gamma = E_\gamma = 273.5\text{ MeV}$

c) $K^0 \rightarrow \pi^-\pi^+ \Rightarrow E_{\pi^-} = E_{\pi^+} = 248.8\text{MeV}$

d) $\Delta^+ \rightarrow p\pi^0 \Rightarrow E_p = 965.9\text{ MeV}$

$E_{\pi^0} = 266.1\text{ MeV}$

e) $\Omega^- \rightarrow \Lambda K^+ \Rightarrow E_\Lambda = 1135.5\text{ MeV}$

$E_{K^+} = 536.7\text{ MeV}$

Question #3

$\pi^+ + p \rightarrow K^+ + K^- + \pi^+ + p$

The invariant mass of the K^+K^- system is given by:

$s^2 = m^2 = (p_+ + p_-)^2$

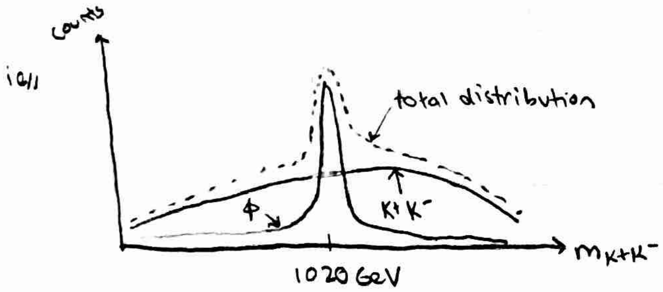
$m^2 = m_+^2 + m_-^2 + 2E_+E_- - 2\vec{p}_+\vec{p}_-$

$m^2 = 2m_K^2 + 2E_+E_- - 2|\vec{p}_+||\vec{p}_-|\cos\theta$

$m^2 = 2(493\text{ MeV})^2 + 2(10\text{ GeV})(5\text{ GeV}) - 2(10\text{ GeV})(5\text{ GeV})\cos 6.3°$

$\Rightarrow m \cong 1.04\text{ GeV}$

If we did observe a single event with an invariant mass of 1020 MeV, this would not be enough to persuade us that a ϕ meson with mass 1020 MeV exists. This is because a K^+K^- system could in fact have this invariant mass without the intermediate step of a ϕ. What we need is to find an excess of events centred around 1020 MeV, producing a peak as discussed in class & tutorial.



[approximate drawing of distribution]

Question #4

$K^- + p \rightarrow \Omega^- + K^+ + K^0$ where K^0 decays with $\tau = 0.8954 \times 10^{-10}$ s

a) The decay of the K^0_S is caused by the weak interaction - we can say this because of its relatively long lifetime. Typically, a strong interaction has a lifetime $< 10^{-23}$ s. You can also see that this is a weak decay since the decay violates strangeness.

Question #4 continued ...

b) In the LAB frame: $\tau = \gamma \tau_0$ where $\gamma = \sqrt{1+(P/m)^2}$

$$= \sqrt{1+(P/m)^2}\,\tau_0$$
$$= \sqrt{1+(5\,\text{GeV}/497.7\,\text{MeV})^2}\,(0.8954\times10^{-10}\,s)$$
$$= 9.0\times10^{-10}\,s$$

$$l = \beta c \tau$$
$$= (P/\gamma m)\,c\tau$$
$$= (P/\gamma m)\,c(\gamma\tau_0)$$
$$= (P/m)\,c\tau_0$$
$$\cong 0.27\,m$$

c) For the threshold kinetic energy, we first need to find just the threshold energy:

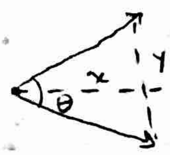
$$E_{TH} = (M^2 - m_K^2 - m_p^2)/(2m_p)$$
$$= ((m_\Omega + m_{K^+} + m_{K^-})^2 - m_K^2 - m_p^2)/(2m_p)$$
$$= [(1672\,\text{MeV} + 493.65\,\text{MeV} + 493.65\,\text{MeV})^2 - (493.65\,\text{MeV})^2 - (938.27\,\text{MeV})^2]/2(938.27\,\text{MeV})$$
$$\cong 3.2\,\text{GeV}$$

The kinetic energy of the incident K^- is $\therefore$ $E_K = E_{TH} - E_{mass}$

$$\cong 3.2\,\text{GeV} - 493.65\,\text{MeV}$$
$$\cong 2.7\,\text{GeV}$$

d) $K^0 \rightarrow \pi^0 + \pi^0$

By conservation of 3-momentum, $\vec{p}_{K^0} = \vec{p}_{\pi^0} + \vec{p}_{\pi^0}$ $\Big\}$ in $x \Rightarrow \vec{p}_{\pi^0 x} = \frac{1}{2}\vec{p}_{K^0}$; in $y \Rightarrow \vec{p}_{\pi^0 y} = \vec{p}_{\pi^0 y}$



For $\vec{p}_{\pi^0 y}$, use rest frame of K_0, where $E_{\pi^0} = \frac{1}{2} m_{K^0}$

Then, $\vec{p}^2 = E^2 - m^2$

$$\vec{p} = \sqrt{(\tfrac{1}{2}m_{K^0})^2 - (m_{\pi^0})^2}$$
$$= 209.05\,\text{MeV}$$

Question #4 continued . . .

Then for $\vec{p}_{\pi^0 x}$, $p_{K^0} = \gamma m_K v_K$

$$= \gamma m_K 0.8c$$

$$= 663.6 \text{ MeV}$$

$$\Rightarrow \vec{p}_{\pi^0 x} = \tfrac{1}{2} p_{K^0}$$

$$= 331.8 \text{ MeV}$$

Then to find the angle: $\theta = \tan(\vec{p}_y / \vec{p}_x)$

$$= \tan(209.05 / 331.8)$$

$$\approx 0.56 \text{ rad}$$

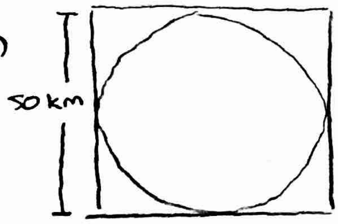
$$\approx 32.2^\circ$$

note: in terms of masses,

$$\theta = \tan^{-1}\left[\frac{\sqrt{(\frac{1}{2} m_{K^0})^2 - (m_{\pi^0})^2}}{\frac{1}{2}\gamma m_{K^0} 0.8c}\right]$$

Question #5

a) To produce muons, you hit a fixed target with high energy protons. This produces pions and kaons, which then decay to muons and neutrinos.

b) 

$r = 50\text{km}/2 = 25 \text{ km}$

$\vec{B} = 5\text{T}$

Recall that for a circular collider: $\vec{F} = q\vec{v} \times \vec{B} = \gamma m v^2/R$

$$qvB = \gamma m v^2/R$$

$$qB = \gamma m v/R$$

$$qB = \vec{p}/R$$

$$\Rightarrow \vec{p} = qBR$$

$$= e(5\text{T})(25 \text{ km})$$

$$= 37.5 \text{ TeV}$$

note: $E \sim \vec{p}$

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Question #5 con't . . .

c) Muons with stationary protons:

$$E_{cm}^2 = (p_\mu + p_p)^2$$
$$E_{cm}^2 = m_\mu^2 + m_p^2 + 2E_\mu m_p$$
$$E_{cm}^2 \cong 2E_\mu m_p$$
$$\Rightarrow E_\mu = E_{cm}^2/2m_p$$
$$= (2\cdot 37.5\,\text{TeV})/2(938.27\,\text{MeV})$$
$$\cong 3.0\times 10^6\,\text{TeV}$$

If the muons collide with atomic electrons rather than protons,

$$\Rightarrow E_{cm} = \sqrt{2E_\mu m_e}$$
$$= \sqrt{2(3.0\times 10^6\,\text{TeV})(511\,\text{keV})}$$
$$\cong 1.75\,\text{TeV}$$

d) Synchrotron radiation, energy loss is proportional to $E_{loss} \propto \frac{4\pi e^2}{R}\left(\frac{E}{mc^2}\right)^4 \propto \left(\frac{1}{m^4}\right)$

Then, relative to electrons,

$$\Delta E_p/\Delta E_e = (m_e/m_p)^4 \sim 8.9\times 10^{-14} \qquad (\times 10^{-12})$$
$$\Delta E_e/\Delta E_e = (m_e/m_e)^4 \sim 1 \qquad (14.8)$$
$$\Delta E_\mu/\Delta E_e = (m_e/m_\mu)^4 \sim 5.5\times 10^{-10} \qquad (\times 10^{-9})$$

Electrons clearly lose much more energy to synchrotron radiation than protons or muons!

Question #6

Since $E = \gamma mc^2 \Rightarrow \gamma = (E/mc^2)$
$\Rightarrow \gamma m = (E/c^2)$

Also: $d\vec{p}/dt = \vec{f}$
$dE/dt = \vec{f}\cdot\vec{v}$

Then we can write

$$d\vec{p}/dt = \frac{d}{dt}(\gamma m\vec{v})$$
$$= m\vec{v}\, d\gamma/dt + \gamma m\, d\vec{v}/dt$$
$$= m\vec{v}\, \frac{d}{dt}(E/mc^2) + (E/c^2)\, d\vec{v}/dt$$
$$= (\vec{v}/c^2)\, dE/dt + (E/c^2)\, d\vec{v}/dt$$

From the expression for $d\vec{p}/dt$; $d\vec{p}/dt = \vec{f} = (\vec{v}/c^2)(\vec{f}\cdot\vec{v}) + (E/c^2)\, d\vec{v}/dt$

$$\vec{f} = (\vec{v}/c^2)(\vec{f}\cdot\vec{v}) + \gamma m\, d\vec{v}/dt$$
$$\Rightarrow m\, d\vec{v}/dt = \left[\vec{f} - \tfrac{1}{c^2}(\vec{v}(\vec{v}\cdot\vec{f}))\right]\gamma^{-1}$$

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Question #6 con't...

a) $\vec{p} = 500$ GeV so that $\gamma = \sqrt{1 + (\vec{p}/m)^2}$

$$= \sqrt{1 + (500\,\text{GeV}/511\,\text{keV})^2}$$
$$= 9.78 \times 10^5$$

Then, $\vec{p} = \gamma m \vec{v} \Rightarrow \vec{v} = \vec{p}/\gamma m$

$$= (500\,\text{GeV})/(9.78\times10^5)(511\,\text{keV})$$

$\cong 1$ $\therefore \vec{v} \sim c$ since I was working in natural units.

b) The electrons feel a constant force of $e\vec{E}$ $\therefore$ $E = \int F dx$

$$E = \int e\vec{E}\,dx$$
$$E = e\vec{E}\Delta d$$

Since $m_e \ll E$, $\vec{p} \sim E$ and we can write: $500\text{ GeV} = e\vec{E}(15\text{ km})$

$$\Rightarrow \vec{E} = 3.33 \times 10^7\ \text{V/m}$$

c) If Newtonian mechanics applied, we have $\Delta E = Fd$

$$\Rightarrow d = \Delta E/F$$
$$d = \tfrac{1}{2}mv^2/e\vec{E}$$
$$= \tfrac{1}{2}(511\text{keV}/c^2)\,c^2/e(3.33\times10^7\ \text{V/m})$$
$$= 7.7\times10^{-3}\ \text{m}$$

d) The length of the accelerator in the electrons rest frame is the proper length

$$d = d_{LAB}/\gamma$$
$$= (15\text{ km})/(9.78\times10^5)$$
$$= 0.015\text{ m}$$