FERMI GAS MODEL — QUANTUM MECHANICS

SIMPLE LIQUID DROP IS INADEQUATE

- NO EXPLANATION OF FACT THAT NUCLEI WITH $N=Z$ (C, O, N) ARE MOST STABLE

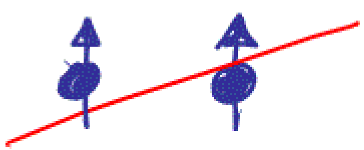
- NO EXPLANATION OF FACT
  - n p → EVEN - EVEN VERY STABLE
  - ODD - ODD UNSTABLE - RARE

- FOR FIXED A, $Z=0$ IS MINIMUM ENERGY → MOST STABLE $p \to n e^+ \nu_e$
  - NO ATOMS! (:-( )

→ QUANTUM STATISTICS

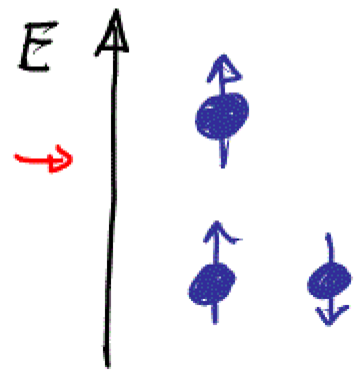
"IF CORRECT FOR COULOMB FORCE, THEN NEUTRON-NEUTRON & PROTON-PROTON BINDING ENERGY IS SAME"

THIS IGNORES PAULI EXCLUSION PRINCIPLE IN AN ATOM

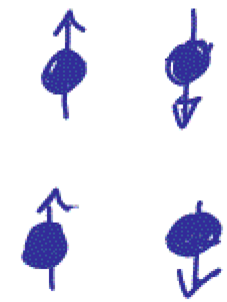


ELECTRONS IN ATOMIC ORBIT

→ PAULI →



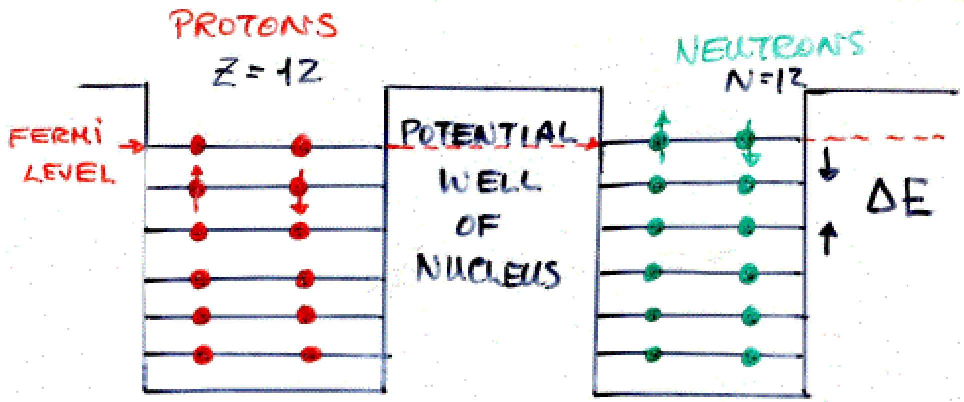
→



FERMI DIRAC → ELECTRONS IN SUCCESSIVE SHELLS → CHEMICAL PERIODIC TABLE

- n, p ARE SPIN $1/2$ → SAME BEHAVIOUR IN NUCLEI
- n, p NOT IDENTICAL, SO DUE TO PAULI SEE DIFFERENT POTENTIALS

FERMI GAS

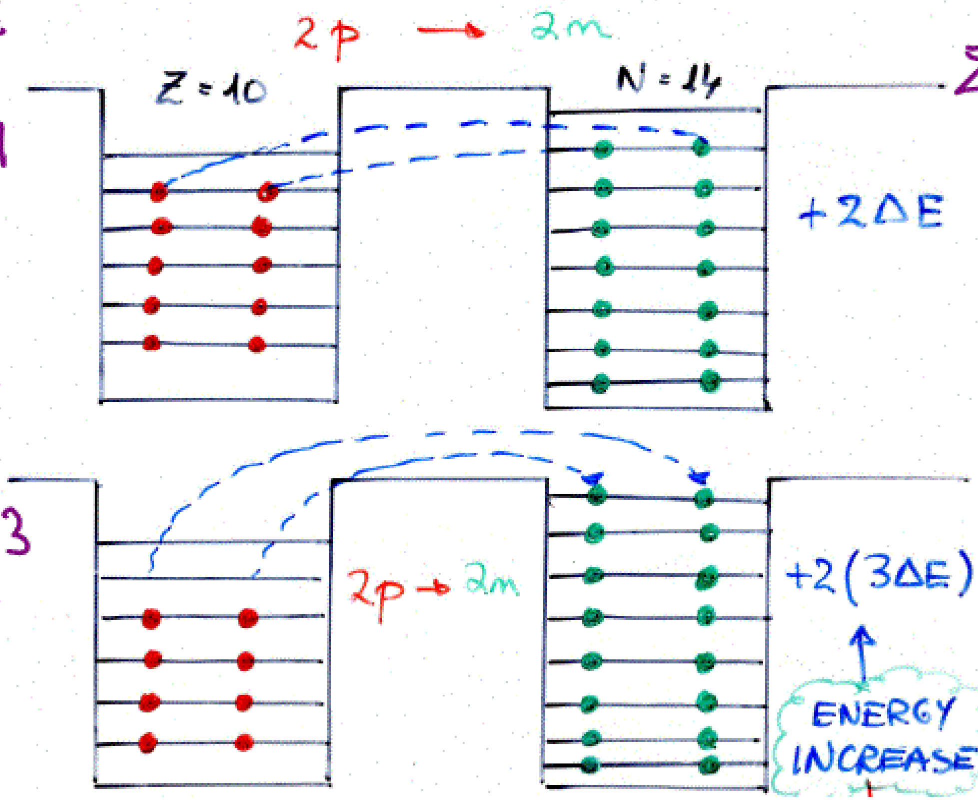


MOVING AWAY FROM

Z = N

BINDING ENERGY REDUCED

ΔE	ΣΔE
1, 1	1, 2
3, 3	5, 8



PROTON FERMI LEVEL

↑

DIVERGE

↓

NEUTRON FERMI LEVEL

ENERGY INCREASE → BINDING ENERGY DECREASE

• AS n, p ASYMMETRY GROWS BY ONE NUCLEON AT A TIME.

BINDING ENERGY DECREASES BY

ΔE	1	2	5	8	13	18	27
$(N-Z)$	2	4	6	8	10	12	14

SO TO GO FROM NUCLEUS $N-Z=0$

TO ONE WITH $N>Z$ WITH $A=N+Z=$ CONSTANT

BINDING ENERGY WILL DECREASE BY

$$\delta E_{ASY} \sim (N-Z)^2 \cdot \frac{\Delta E}{8}$$

MODIFY "LIQUID DROP"

$$-a_s \frac{(Z-N)^2}{A}$$

NUMBER OF ENERGY LEVELS

$\propto$ VOLUME OF POTENTIAL WELL

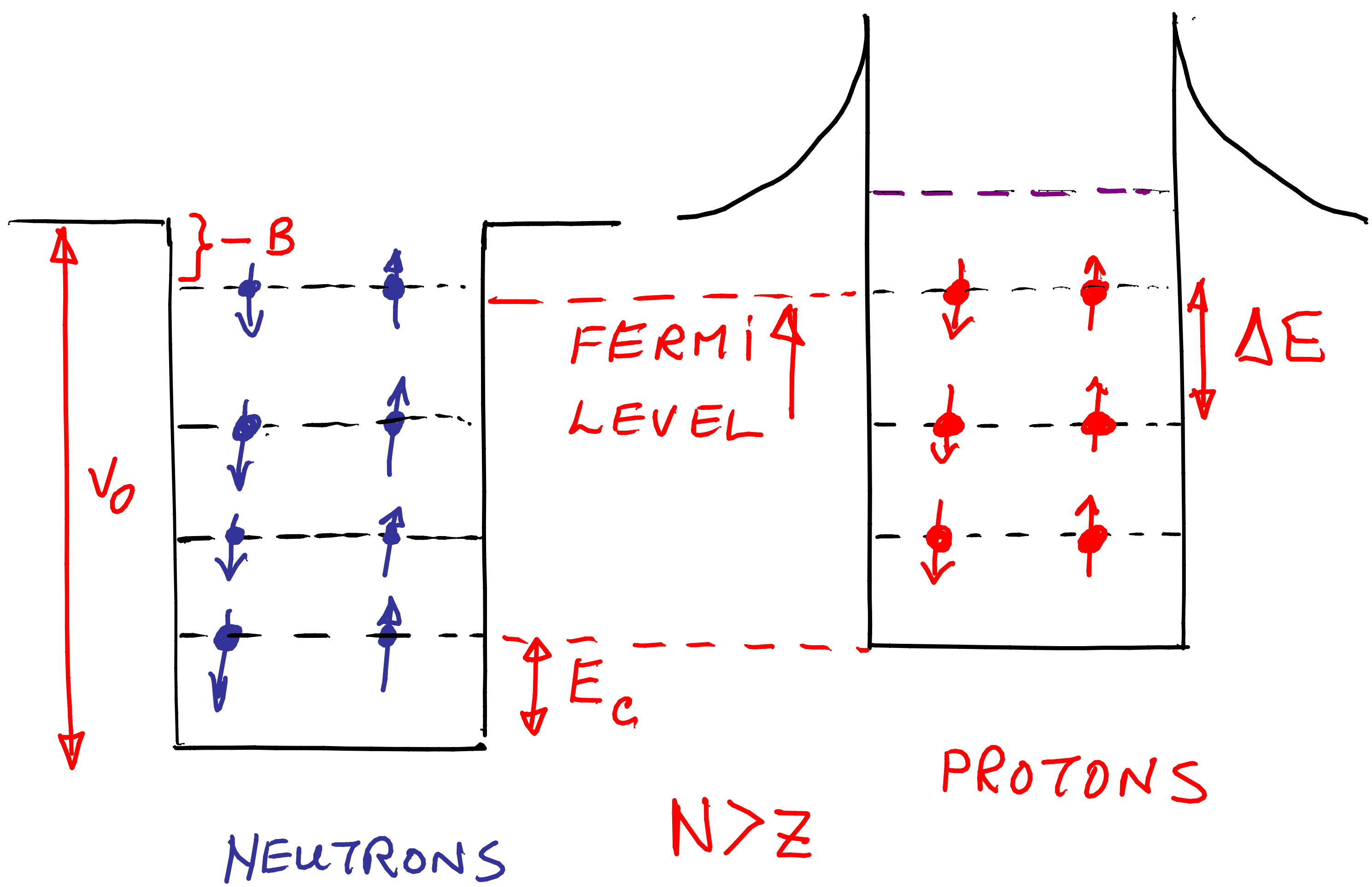
$\propto A$

$\Delta E \propto 1/A$

MORE ENERGY LEVELS

SMALLER SPACING.

Fermi Gas Model



NEUTRON AND PROTON POTENTIAL WELL DEPTHS $\longrightarrow$ DIFFERENT.

IF THEY WERE THE SAME IN HEAVY NUCLEI WHERE $N > Z \rightarrow E_F^{NEUTRON} > E_F^{PROTON}$

THEN BINDING ENERGY OF LAST NUCLEON WOULD BE CHARGE DEPENDANT

$\hookrightarrow$ NOT TRUE
CHARGE INDEPENDENCE

- ALSO IF $E_F^N > E_F^P$ NEUTRONS WOULD BE ENERGETICALLY FAVOURED TO DECAY TO PROTONS VIA β-DECAY $\rightarrow$ PROTONS

$\rightarrow N > Z$ UNSTABLE $\rightarrow$ NOT TRUE

NON-RELATIVISTICALLY, THE FERMI ENERGY

$$E_F = \frac{p_F^2}{2M}$$ NUCLEON MASS

VOLUME IN MOMENTUM SPACE

$$V_{p_F} = \frac{4\pi}{3} p_F^3$$

PHASE SPACE = CONFIGURATION SPACE × MOMENTUM SPACE

$$V_{TOT} = \frac{4\pi}{3} r_0^3 A \cdot \frac{4\pi}{3} p_F^3$$

$$= \left(\frac{4\pi}{3}\right)^2 A \cdot (r_0 p_F)^3$$

$\propto$ NUMBER OF QUANTUM STATES OF SYSTEM

NUMBER OF FERMIONS THAT CAN FILL STATES UP TO THE FERMI LEVEL

$$n_F = \frac{2\, V_{TOT}}{(2\pi\hbar)^3} = \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar}\right)^3$$

SPIN UP/DOWN

FOR $N = Z = A/2 \longrightarrow$ ALL STATES FILLED

$$N = Z = A/2 = 4/9\pi \cdot A \left(\frac{r_0 p_F}{\hbar}\right)^3$$

$$p_F = \frac{\hbar}{r_0}\left(\frac{9\pi}{8}\right)^{1/3}$$

DOES NOT DEPEND ON NUMBER OF NUCLEONS

$$p_F = \frac{\hbar}{r_0}\left(\frac{9\pi}{8}\right)^{1/3}$$

$$E_F = \frac{p_F^2}{2M} = \frac{1}{2M}\left(\frac{\hbar}{r_0}\right)^2\left(\frac{9\pi}{8}\right)^{2/3} \sim 33\ \mathrm{MeV}$$

NOW $B/A \sim -8\ \mathrm{MeV}$

SO THE DEPTH OF THE POTENTIAL WELL

$$V_0 = E_F + B \approx 40\ \mathrm{MeV}$$

THE NUCLEONS ARE NON-RELATIVISTIC

AVERAGE KINETIC ENERGY

$$\langle E \rangle = \frac{\int_0^{p_F} E \, d^3p}{\int_0^{p_F} d^3p}$$

$$= \frac{\int_0^{p_F} \frac{p^2}{2M} \cdot p^2 \sin\theta \, d\theta \, d\phi \, dp}{\int_0^{p_F} p^2 \sin\theta \, d\theta \, d\phi \, dp}$$

$$= \frac{p_F^5}{5} \frac{1}{2M} \cdot \frac{3}{p_F^3} = \frac{3}{5} \frac{p_F^2}{2M}$$

$$\approx 24 \, \text{MeV}$$

TOTAL AVERAGE KINETIC ENERGY

$$\left\langle E(Z,N) \right\rangle = N\langle E_N \rangle + Z\langle E_Z \rangle$$

$$= \frac{3}{10M}\left(N p_N^2 + Z p_Z^2\right)$$

USE $\quad N = Z = \frac{A}{2} = \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar}\right)^3$

$$\langle E(Z,N) \rangle = \frac{3}{10M} \frac{\hbar^2}{r_0^2} \left(\frac{9\pi}{4}\right)^{2/3} \frac{N^{5/3} + Z^{5/3}}{A^{2/3}}$$

FOR A GIVE VALUE OF A

$\langle E(Z,N) \rangle$ MINIMUM WHEN $N = Z = \frac{A}{2}$

LOOK AT BEHAVIOUR AROUND MINIMUM

PUT $Z - N = \varepsilon$, $Z + N = A$ FIXED

$$Z = \frac{A}{2}\left(1 + \varepsilon/A\right), \quad N = \frac{A}{2}\left(1 - \varepsilon/A\right)$$

IF ASSUME $\varepsilon/A << 1$

$$Z^{5/3} = \left(\frac{A}{2}\right)^{5/3}\left(1 + \frac{\varepsilon}{A}\right)^{5/3} = \frac{A^{5/3}}{2}\left(1 + \frac{5}{3}A + \frac{5/3(5/3-1)}{2}A^2 + \ldots\ldots\right.$$

$$N^{5/3} = \left(\frac{A}{2}\right)^{5/3}\left(1 - 5/3\,A + \frac{5/3(5/3-1)A^2}{2} + \ldots\right)$$

INSERT THESE EXPASIONS INTO

$$\langle E(Z,N)\rangle = \frac{3}{10M}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\left(\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}\right)$$

AND PUT $\varepsilon \to N-Z$

$$\langle E(Z,N)\rangle = \frac{3}{10M}\frac{\hbar^2}{r_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A+\frac{5}{9}\frac{(Z-N)^2}{A}\right)$$

ASYMMETRY TERM — $\frac{\langle E\rangle}{A} \propto \frac{(Z-N)^2}{A^2}$

EXPERIMENTALLY

pp
nn

MORE TIGHTLY BOUND THAN

np

MORE STABLE

PAIRING TERM

$A_{ODD} \rightarrow$ Z_{EVEN} N_{ODD} OR Z_{ODD} N_{EVEN} } NO EFFECT

$A_{EVEN} \rightarrow$ Z_{EVEN} N_{EVEN} { MORE STABLE } Z_{ODD} N_{ODD}

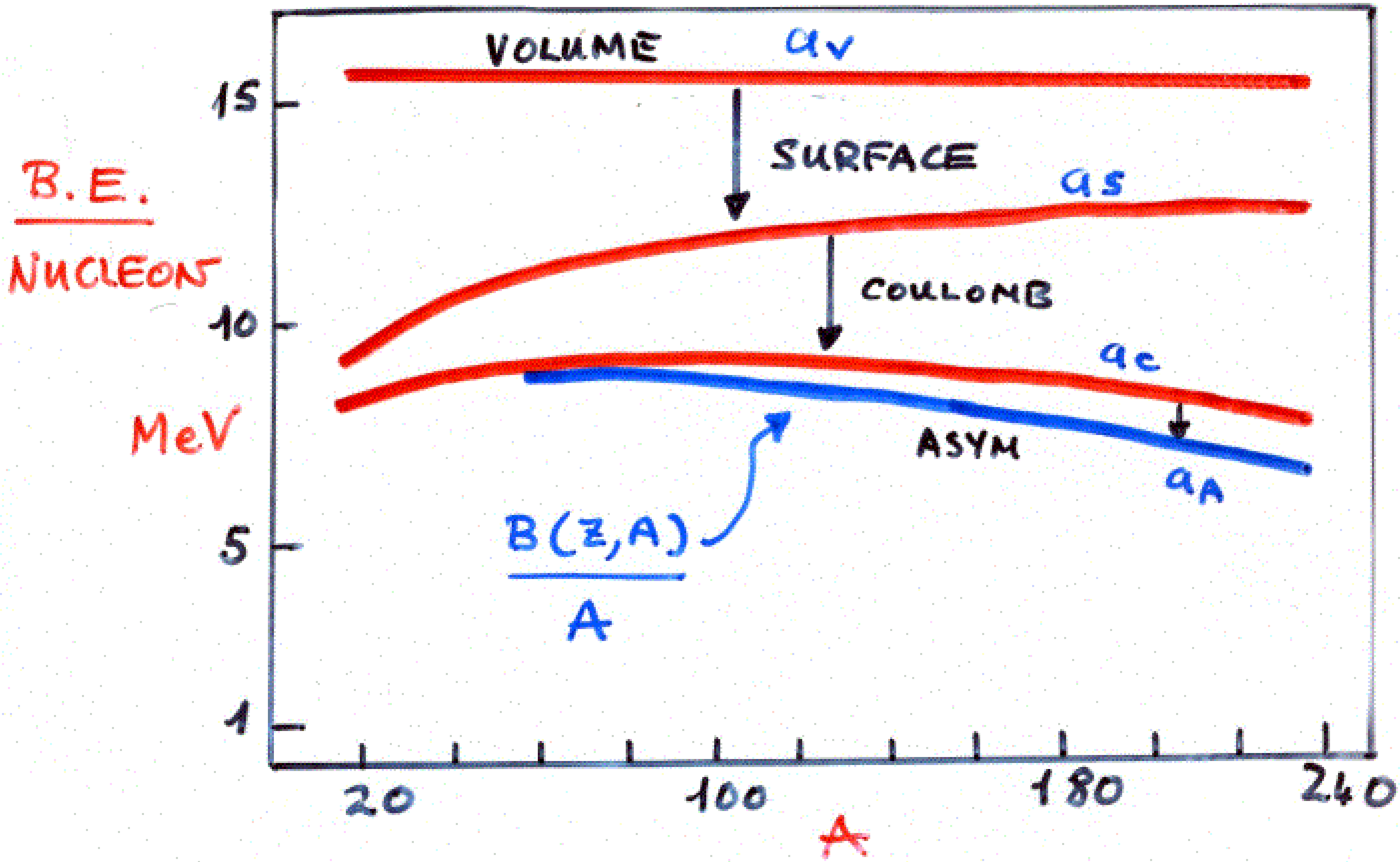
LIQUID DROP PAIRING TERM

$$\delta(Z, A) = \frac{a_P}{A^{1/2}}$$

$a_P \sim 12\,\mathrm{MeV}$

EVEN-EVEN +VE, ODD-ODD −VE

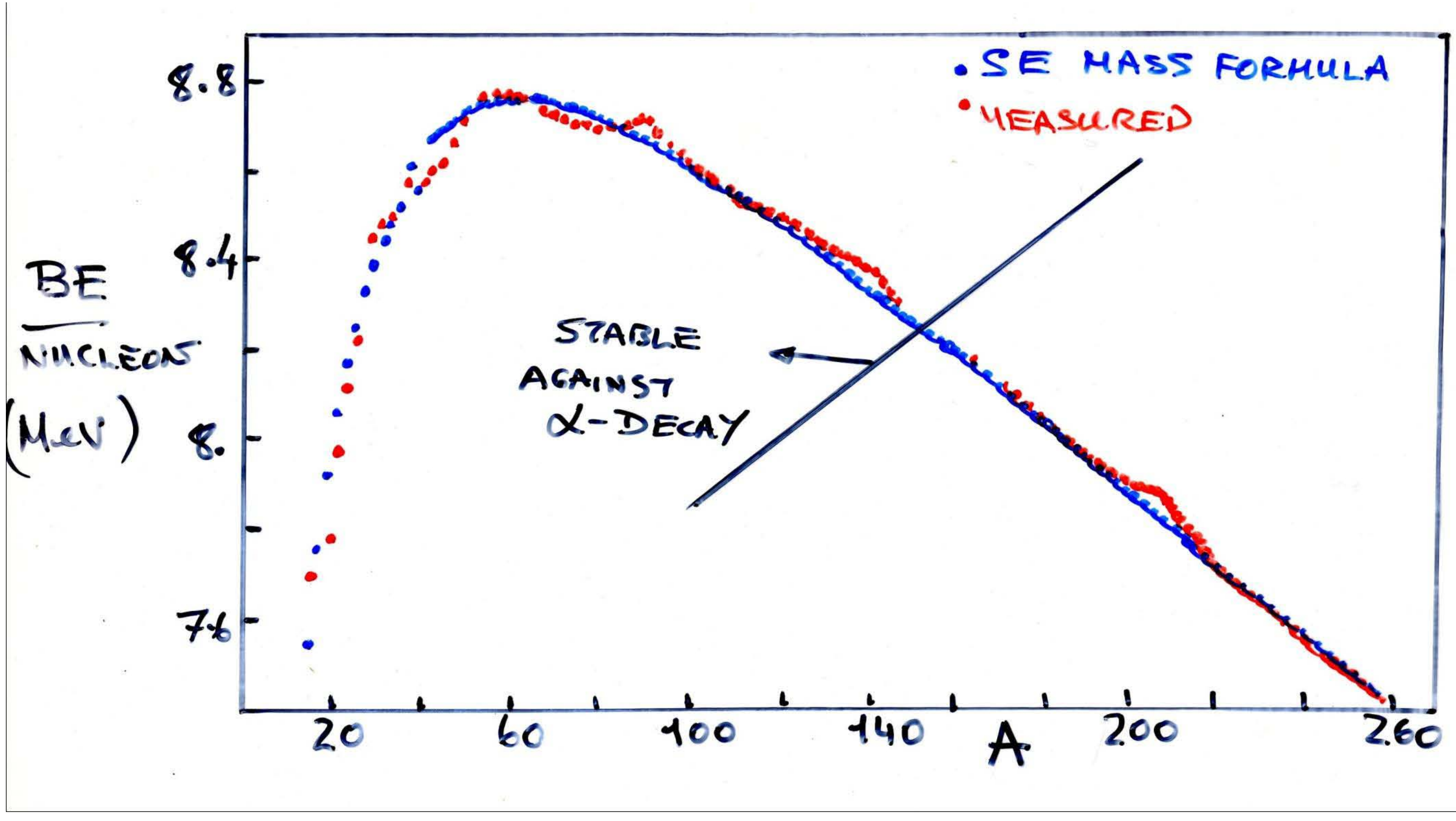
SEMI-EMPIRICAL (BETHE-WEIZSÄCKER) FORMULA



$$B = a_V A - a_S A^{2/3} - \frac{a_C Z^2}{A^{1/3}} - \frac{a_A (A-2Z)^2}{A} \pm \delta(Z,A)$$

VOLUME SURFACE COULOMB ASYMMETRY PAIRING

a_i = 15MeV 17 .7 23 12



SE MASS FORMULA
MEASURED
BE / NUCLEON (MeV)
8.8
8.4
8.
7.6
STABLE AGAINST α-DECAY
20
60
100
140
200
260
A

$M(Z,A)c^2$	Nuclear rest mass energy
$= ZM_pc^2 + (A-Z)M_nc^2 - B(Z,A)$	Rest of mass of constituents less the binding energy

where

$B(Z,A) =$	$+a_V A$	Volume binding term
	$-a_S A^{2/3}$	Surface energy term
	$-a_C Z^2/A^{1/3}$	Coulomb term
	$-a_A (A-2Z)^2/A$	Asymmetry term
	$\left\{\begin{array}{ll} -a_P/A^{1/2} & \text{oo nuclei} \\ +0 & \text{eo and oe nuclei} \\ +a_P/A^{1/2} & \text{ee nuclei} \end{array}\right\}$	Pairing term

M_pc^2 = rest mass energy of the proton = 938.280 MeV.
M_nc^2 = rest mass energy of the neutron = 939.573 MeV.
A favoured set of values for the coefficients:

$$\begin{aligned} a_V &= 15.56 \text{ MeV},\\ a_S &= 17.23 \text{ MeV},\\ a_C &= 0.697 \text{ MeV},\\ a_A &= 23.285 \text{ MeV},\\ a_P &= 12.0 \text{ MeV}. \end{aligned}$$

To obtain the atomic rest mass energy, change M_P, the proton mass, to M_H, the mass of the hydrogen atom, thus:

$$\mathcal{M}(Z,A)c^2 = \text{atomic rest mass energy}$$
$$\approx ZM_Hc^2 + (A-Z)M_nc^2 - B(Z,A).$$

(Note: $\approx$ because this formula neglects some atomic electron binding energy.)
M_Hc^2 = rest mass energy of the hydrogen atom = 938.791 MeV.

NUCLEAR INSTABILITY

β- DECAY

ISOBAR WITH LARGE SURPLUS OF NEUTRONS

$$n \to p e^- \bar{\nu}_e$$

ISOBARS WITH LARGE SURPLUS OF PROTONS

$$p \to n e^+ \nu_e$$

ATOMIC ELECTRON

ELECTRON CAPTURE

$$p + e^- \to n \, \nu_e$$

α - DECAY

$$(N, Z) \to (N-2, Z-2) + \alpha$$

FISSION

$$(N, Z) \to (N/2, Z/2) + (N/2, Z/2)$$

APPROX EQUAL FRAGMENTS

β - DECAY

USE SEMI-EMPIRICAL MASS FORMULA

REARRANGE TERMS

$$M(Z,A) = \alpha A - \beta Z + \gamma Z^2 + \frac{\delta}{A^{1/2}}$$

$$\alpha = M_n - a_v + a_s/A^{1/3} + a_a/4$$

$$\beta = a_a + (M_n - M_p - m_e)$$

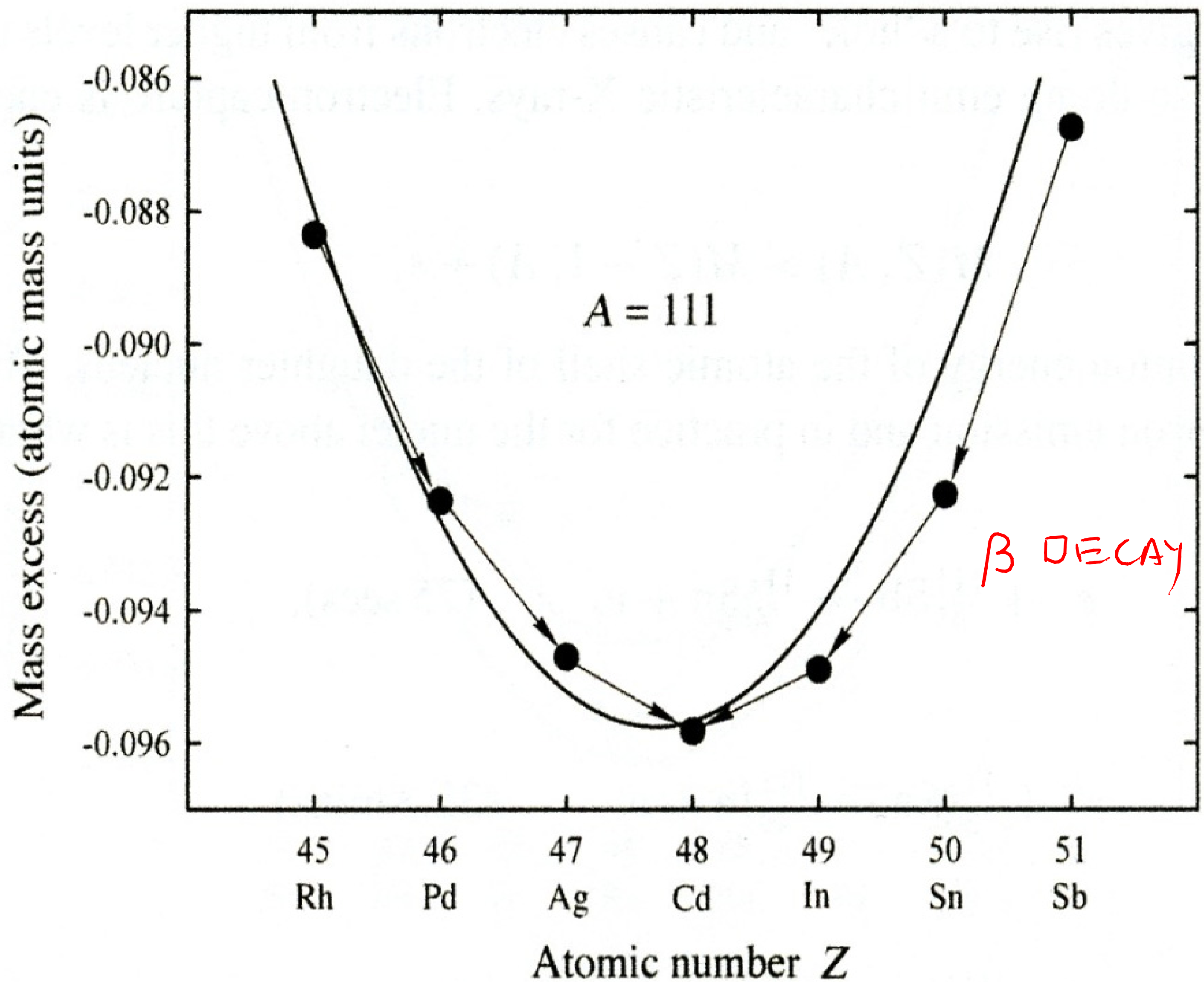
$$\gamma = \frac{a_a}{A} + a_c/A^{1/3}$$

$M(Z,A)$ IS QUADRATIC IN Z AT FIXED A

$$\frac{dM}{dZ} = -\beta + 2Z\gamma$$

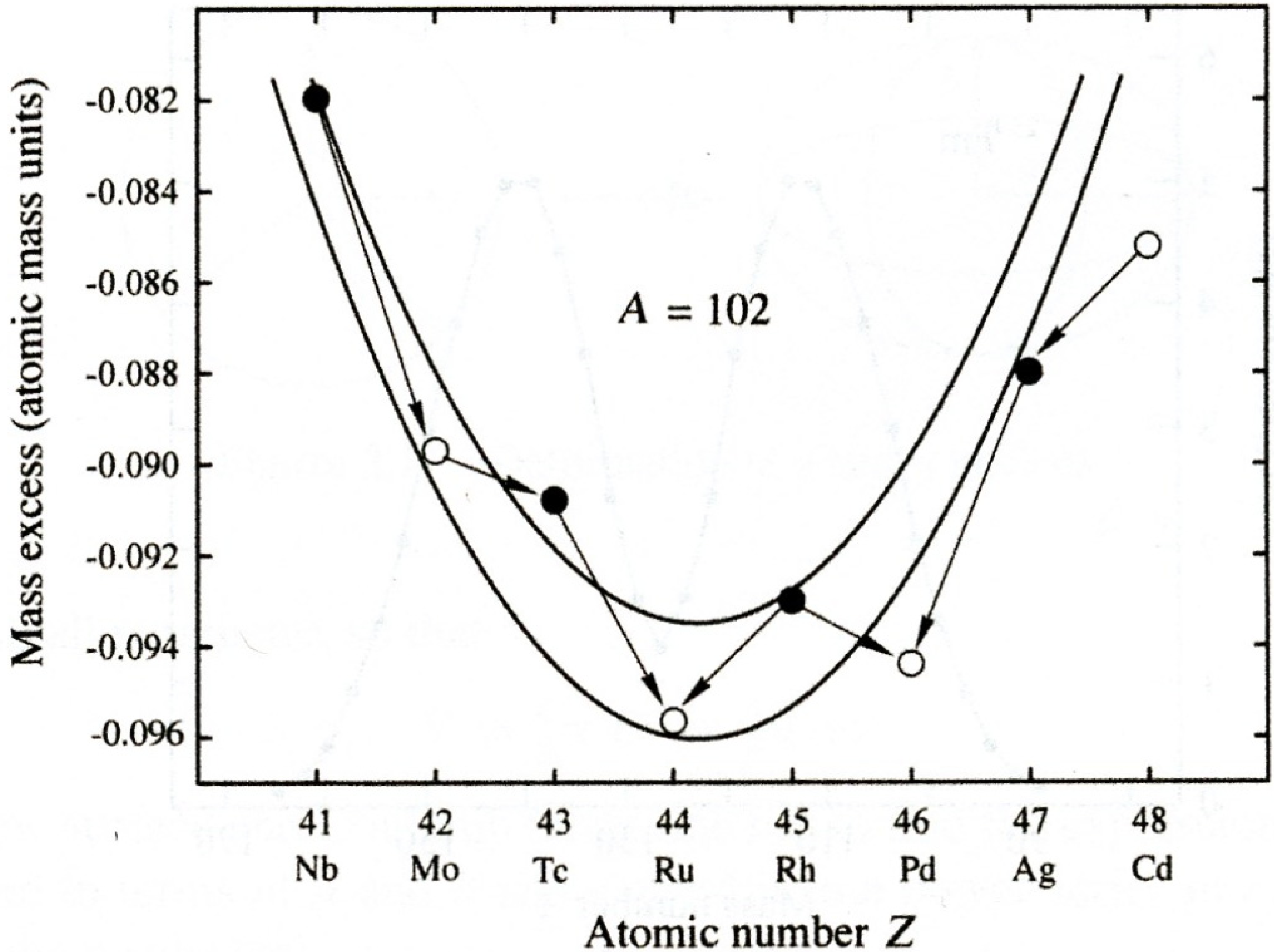
MINIMUM is $Z = \frac{\beta}{2\gamma}$

FOR FIXED A MOST STABLE NUCLEUS



-0.086
-0.088
-0.090
-0.092
-0.094
-0.096
Mass excess (atomic mass units)
A = 111
β DECAY
45 Rh
46 Pd
47 Ag
48 Cd
49 In
50 Sn
51 Sb
Atomic number Z

ODD A NUCLEI



EVEN NUCLEI — PAIRING TERM ±

β EMITTER LIFETIMES 10^{-3} s $\rightarrow$ 10^{16} YEARS

ODD MASS eg $A = 111$

MINIMUM IS AT ${}^{111}_{48}Cd$

ISOBARS WITH MORE NEUTRONS

$$n \rightarrow p\, e^-\, \bar{\nu}_e$$

$$^{111}_{45}Rh \rightarrow {}^{111}_{46}Pb + e^- + \bar{\nu}_e$$

ISOBARS WITH MORE PROTONS

$$p \rightarrow n\, e^+\, \nu_e$$

$$e^- + p \rightarrow n\, \nu_e$$

$$e^- + {}^{111}_{51}Sb \rightarrow {}^{111}_{50}Sn$$

EVEN MASS NUCLEI

EVEN - EVEN
ODD - ODD

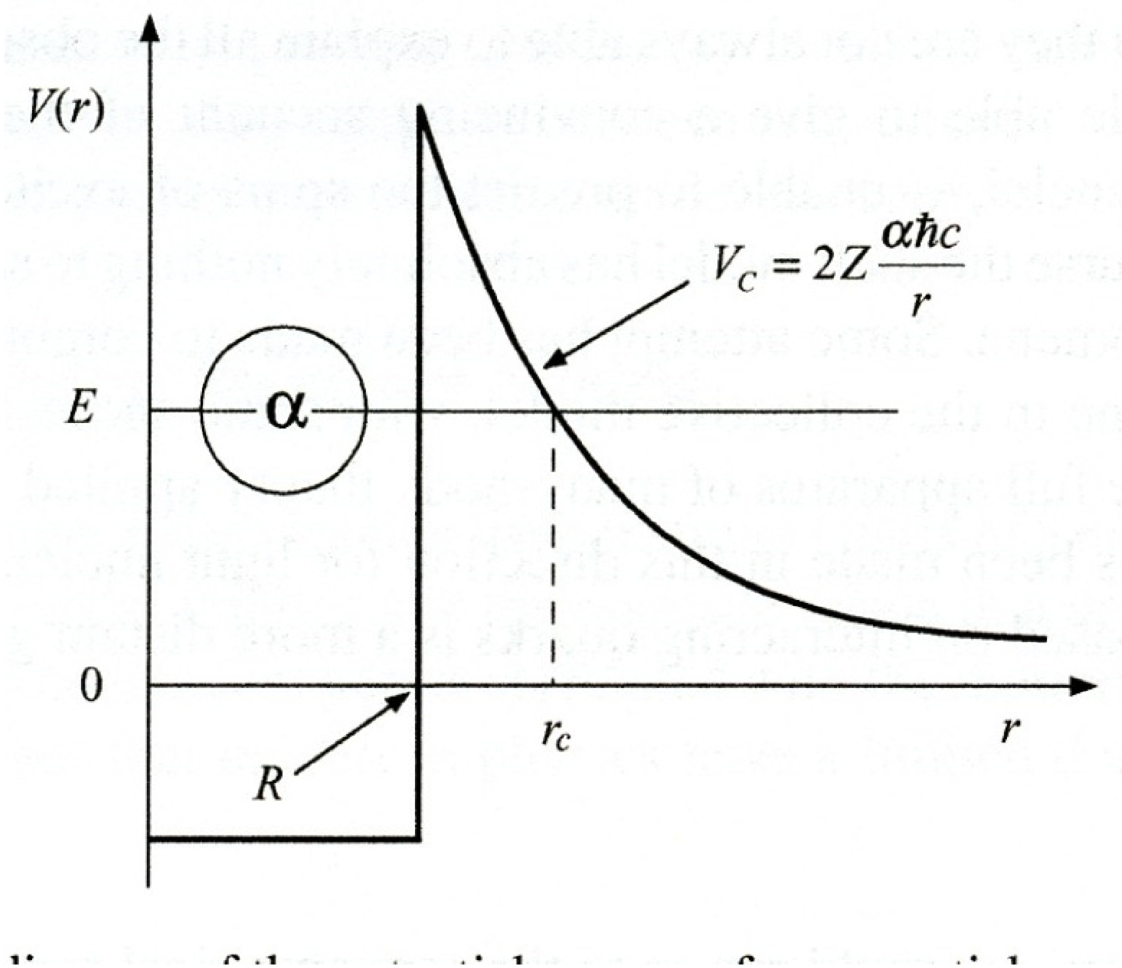
NEARLY ALL STABLE AGAINST β - DECAY

eg $A = 102$ LOWEST ${}^{102}_{44}Ru$ STABLE

${}^{102}_{46}Pb$ IS ALSO STABLE SINCE TWO ODD-ODD NEIGHBORS ARE MORE MASSIVE

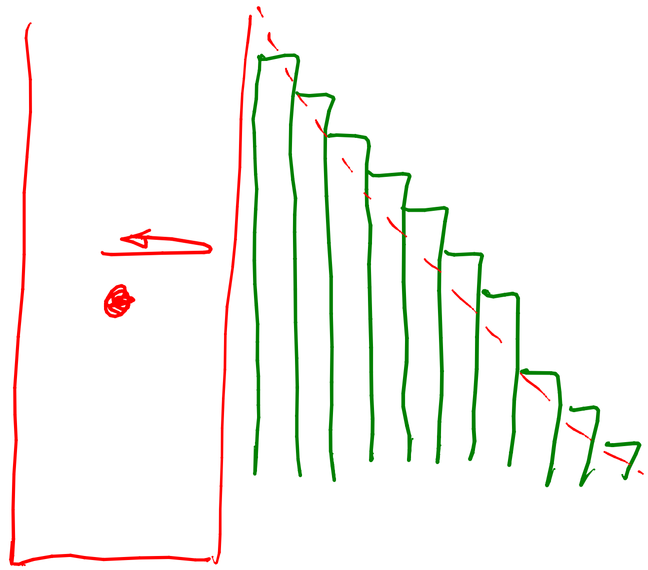
SO $\rightarrow$ TWO STABLE ISOBARS

ALPHA DECAY



α PARTICLE TUNNELS OUT OF NUCLEUS

MODEL BARRIER AS SERIES OF SLICEL



TRANSMISSION PROBABILITY $T \approx e^{-2k\,\Delta r}$

BARRIER HEIGHT

$$\hbar k = \left(2m\,|V_c - E_\alpha|\right)^{1/2}$$

$$T = e^{-G}$$

$$G = \frac{2}{\hbar}\int_{R}^{r_c}\left(2m\,|V_c(r) - E_\alpha|\right)^{1/2}$$

GAMMOV FACTOR

GAMMOW FACTOR $\quad G \approx \dfrac{4\pi \alpha Z}{\beta} \quad \cancel{\beta}\ v/c$

$$\begin{pmatrix}\text{PROB / UNIT TIME}\\ \text{OF TUNNELING}\end{pmatrix} = \begin{pmatrix}\text{PROB OF } \alpha\\ \text{IN NUCLEUS}\end{pmatrix} \times \begin{pmatrix}\text{FREQUENCY}\\ \text{OF COLLISION}\\ \text{WITH BARRIER}\end{pmatrix} \times \begin{pmatrix}\text{PROB OF BARRIER}\\ \text{PENETRATION}\end{pmatrix}$$

$$\lambda = \frac{w(\alpha)\, v_\alpha}{2R}\, e^{-G}$$

$$G \propto \frac{Z}{\beta} \propto \frac{Z}{\sqrt{E_\alpha}} \Rightarrow \log_{10} t_{1/2} = a + b\, Z E^{-\frac{1}{2}}$$

!

GEIGER-NUTTALL

WHY DO LARGE A NUCLEI α-DECAY?

ALLOWED IF $B(2,4) > B(Z,A) - B(Z-2, A-4)$

APPROXIMATE LINE OF STABILITY BY $Z = N$

$\frac{dB}{dA}$ HOW B VARIES IF

$B \to B + dB$

$A \to A + dA$

FOR $A \to A \to A-4$

ENERGY CHANGE $= 4\frac{dB}{dA}$

$$4\,\frac{dB}{dA} = 4\left[A\,\frac{d(B/A)}{dA} + B/A\right]$$

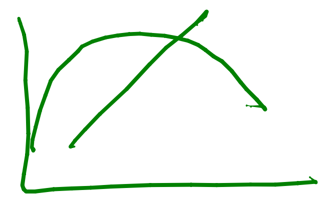
$$\frac{d(B/A)}{dA} = B\,\frac{dA^{-1}}{dA} + A^{-1}\,\frac{dB}{dA}$$

$$= -B/A^2 + \frac{1}{A}\,dB/dA$$

$$\frac{dB}{dA} = \left[\frac{A\,d(B/A)}{dA} + \frac{B}{A}\right]$$

$$\frac{dB}{dA} = \left[A \frac{d(B/A)}{dA} + \frac{B}{A} \right]$$

THIS IS A STRAIGHT LINE IN BINDING ENERGY PLOT



$$\frac{d(B/A)}{dA} \sim 7.7 \times 10^{-3} \text{ MeV FOR } A > 120$$

AND FOR AN ALPHA $B(2,4) = 28.3\,\text{MeV}$

$$28.3 = 4\left(B/A - 7.7\times10^{-3} A\right)$$

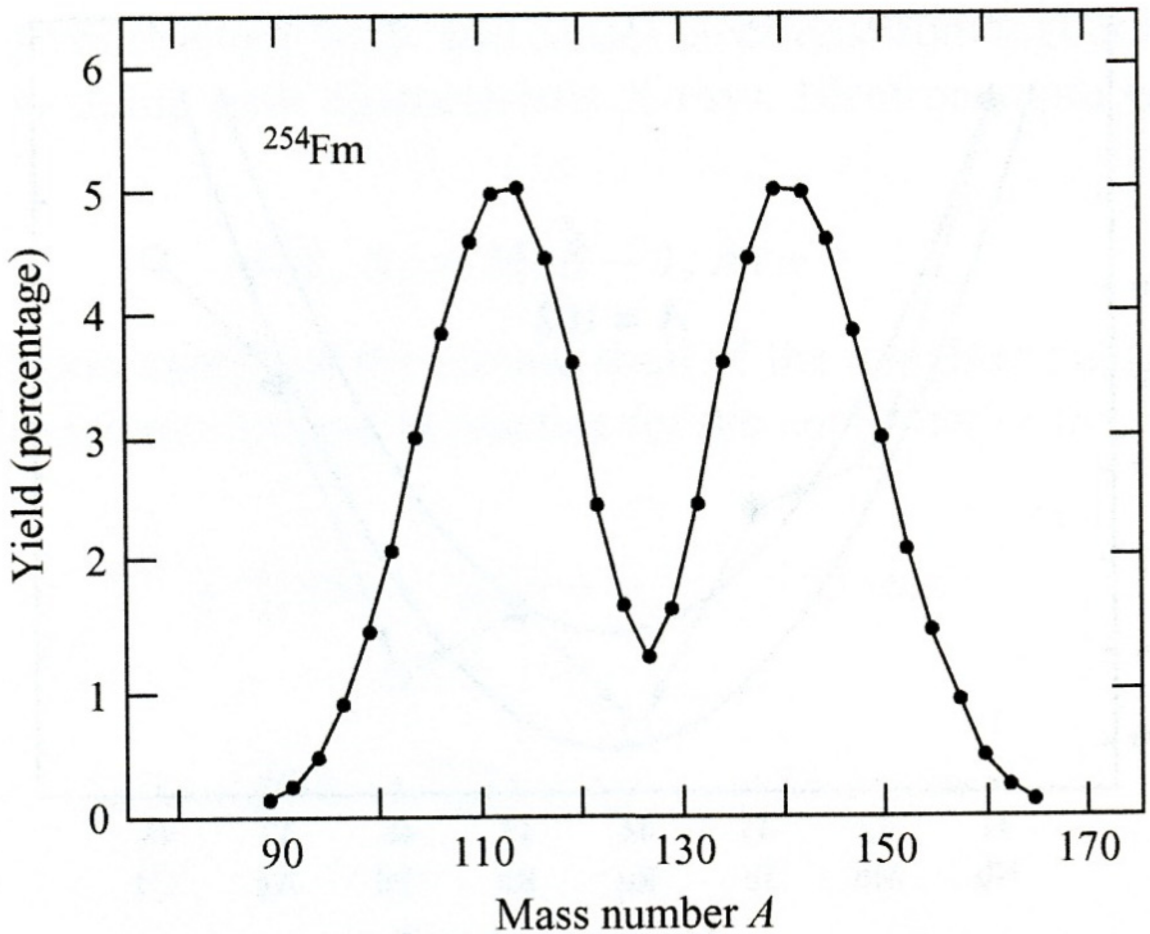
INTERSECTS BE CURVE @ $A \approx 150$

α DECAY DOMINATES FOR $A > 150$

FISSION – NUCLEUS BREAKS INTO TWO EQUAL FRAGMENTS

$$^{238}_{92}U \rightarrow {}^{145}_{57}La + {}^{90}_{35}Br + 3n$$

$\hookrightarrow$ 154 MeV



FISSION $3 \times 10^{-24}\ s^{-1}$

α $5 \times 10^{-18}\ s^{-1}$

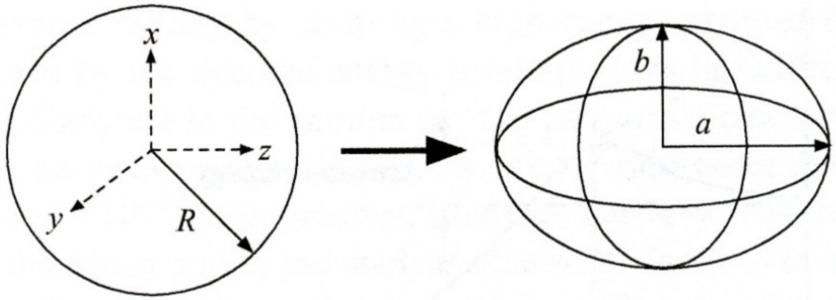


Figure 2.17 Deformation of a heavy nucleus.

$$R = \frac{a}{1+\varepsilon}, \quad R^2 = b^2(1+\varepsilon)$$

$$R^3 = ab^2$$

ELLIPSOID

$$a = R(1+\varepsilon)$$

$$b = R/(1+\varepsilon)^{1/2}$$

$$\text{VOLUME} = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi ab$$

$$\text{SURFACE AREA} = S(\varepsilon) = 4\pi R^2\left(1 + \frac{2}{5}\varepsilon^2 - \cancel{\frac{52}{105}\varepsilon^3}\right)$$

COULOMB ENERGY $= E_C = \frac{\rho^2}{(4\pi\varepsilon_0)^2} \frac{1}{2} \iint \frac{d^3\vec{r}\, d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$

$$= \frac{3}{5} \frac{(Ze)^2}{4\pi\varepsilon_0 R} \cdot \left(1 - \frac{1}{5}\varepsilon^2 + \frac{4}{21}\varepsilon^2\right)$$

SEMIEMPIRICAL MASS FORMULA

$$\left\{ \begin{aligned} E_C &= a_c Z^2 A^{-1/3} \left(1 - \frac{1}{5}\varepsilon^2\right) \\ E_S &= a_s A^{2/3} \left(1 + \frac{2}{5}\varepsilon^2\right) \end{aligned} \right.$$

IF NUCLEUS DEFORMED — E_S, E_C WILL CHANGE

IF DEFORMED NUCLEAS HAS LOWER E

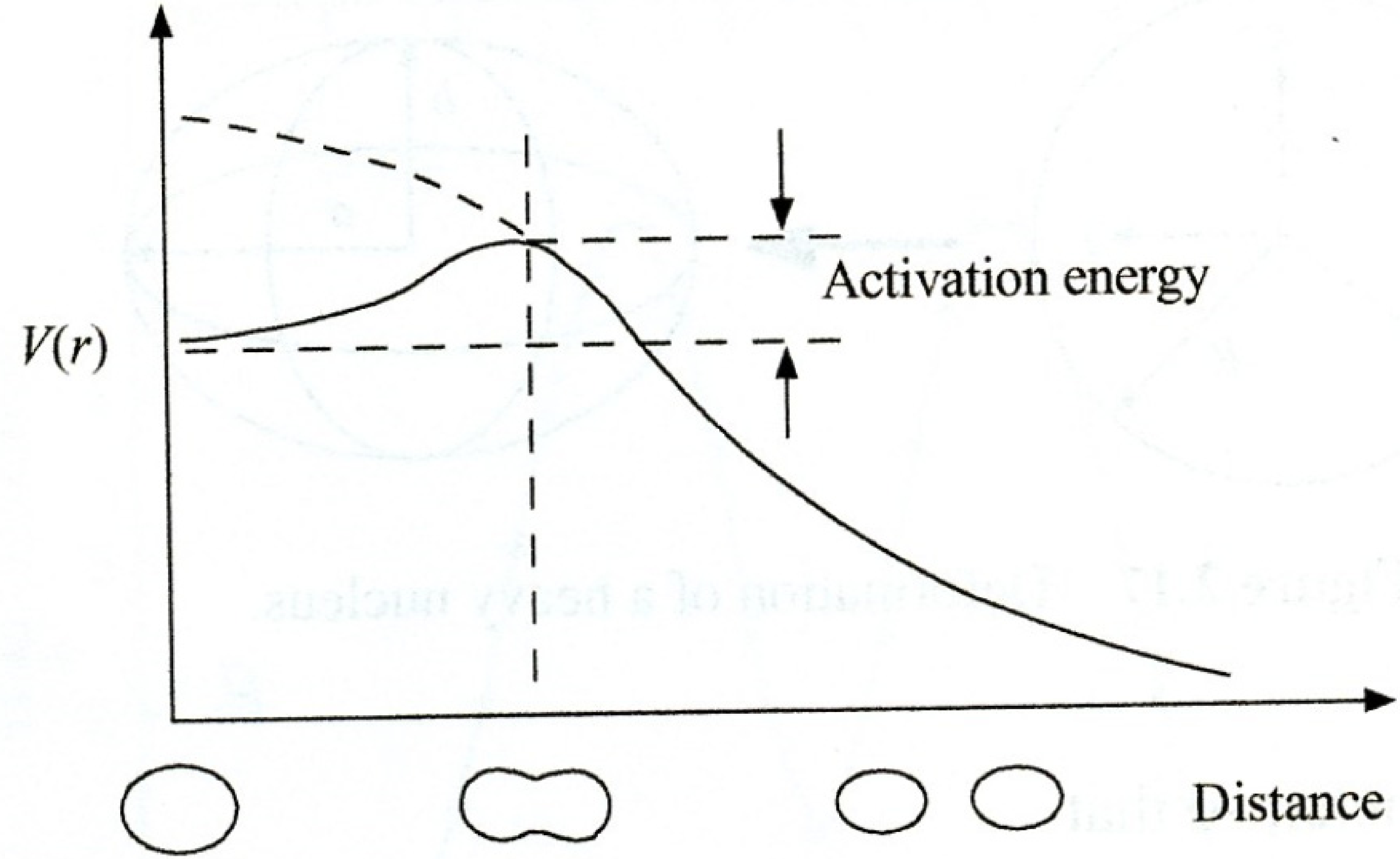
$\rightarrow$ FISSION

$$\Delta E = (E_S + E_C)_{\text{DEFORMED}} - (E_S + E_C)_{\text{SEMI EMPIRICA}}$$

$$= \frac{\varepsilon^2}{5}\left(2 a_s A^{2/3} - a_c Z^2 A^{-1/3}\right)$$

$\Delta E < 0$ FOR $\dfrac{Z^2}{A} \geq \dfrac{2a_s}{a_c} \simeq 49$

$\Delta E < 0$ FOR $Z > 116$, $A > 270$


Activation energy
$V(r)$
Distance