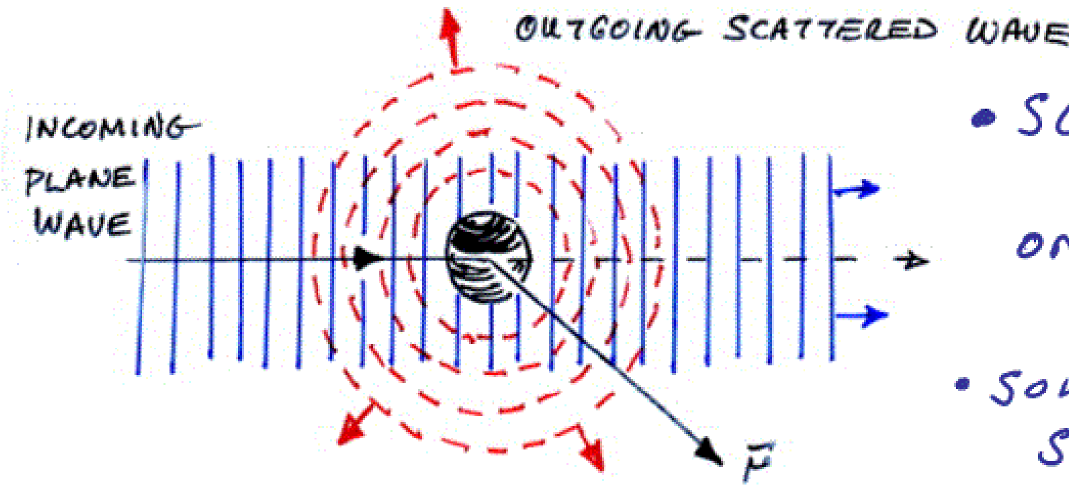
FORM FACTORS – SIZE & SHAPE OF NUCLEI & PARTICLES

SCATTERING EXPERIMENTS

- DETERMINE NUCLEAR SIZE
- DETERMINE NUCLEAR SHAPE
  - DISTRIBUTION OF "MATTER"
  - DISTRIBUTION OF ELECTRIC CHARGE
- DETERMINE SIZE OF PROTON
- DETERMINE SHAPE OF PROTON
  - DISTRIBUTION OF ELECTRIC CHARGE
    - → DIRECT OBSERVATION OF QUARKS

SCATTERING IN NON-RELATIVISTIC QUANTUM MECHANICS

- WE WANT TO GET A FEEL FOR HOW EXPERIMENTS CAN DETERMINE STRUCTURE
- BUILD ON YOUR KNOWLEDGE OF QUANTUM MECHANICS



- SCATTERING IS A WEAK PERTURBATION ON FREELY PROPAGATING PLANE WAVE STATE
- SOLVE TIME DEPENDENT SCHRÖDINGER

$$\left(\nabla^2 + k^2\right)\psi = \frac{2m}{\hbar^2} V \psi$$

- INCOMING $\psi = e^{i\bar{p}\cdot\bar{r}/\hbar}$

- OUTGOING $\psi = e^{ip\cdot\bar{r}/\hbar} + f(\theta,\phi)\dfrac{e^{i\bar{p}\cdot\bar{r}/\hbar}}{|\bar{r}|}$

OUTGOING PLANE WAVE

OUTGOING SPHERICAL WAVE

SOLUTION BY GOLDEN RULE (FERMI)

TRANSITION $|\psi_a\rangle \rightarrow |\psi_b\rangle$ CAUSED BY POTENTIAL V

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \langle \psi_b | V | \psi_a \rangle \right|^2$$

TAKE $|\psi_a\rangle = \exp(i\bar{k}_a \cdot \bar{r})$ $\quad \bar{p} = \hbar \bar{k}$

WHAT ABOUT $|\psi_b\rangle$? SAY V IS SUCH A SMALL PERTURBATION, THAT $|\psi_b\rangle \approx |\psi_a\rangle$

$$|\psi_a\rangle = \exp(i\bar{k}_a \cdot \bar{r}) \rightarrow |\psi_b\rangle = \exp(i\bar{k}_b \cdot \bar{r})$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \langle \psi_b | V | \psi_a \rangle \right|^2 = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int \psi_b^+ V \psi_a \right|^2$$

$$= \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int V e^{i(\bar{k}_a - \bar{k}_b)\cdot\bar{r}} \right|^2$$

NORMALIZATION & DENSITY OF FINAL STATES

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left| \int V e^{i(\bar{k}_a - \bar{k}_b)\cdot\bar{r}} \right|^2$$

INTERACTION POTENTIAL → PHASE CHANGE

MOMENTUM TRANSFER

GENERALLY $\dfrac{d\sigma}{d\Omega} = |f(\bar{q})|^2$ $\quad \bar{q} = \bar{p}_a - \bar{p}_b$

FOR OUR SCATTERING EXAMPLE

MOMENTUM TRANSFER

$$f(\bar{q}) = \frac{-m}{2\pi\hbar^2} \int V(\bar{x})\, e^{i\bar{q}\cdot\bar{x}/\hbar}\, d^3x$$

SCATTERING POTENTIAL

PLANE WAVE STATE

FOR A SPHERICALLY SYMMETRIC POTENTIAL (eg. COULOMB)

$$f(q) = \frac{-m}{2\pi\hbar^2}\int V(\bar{x})\, e^{i\bar{q}\cdot\bar{x}/\hbar}\, d^3x$$

$\int$ OVER 4π $\longrightarrow$ CHANGE VARIABLE TO $z = iqx\cos\theta/\hbar$

$x = |\bar{x}| \qquad q = |\bar{q}|$

$$f(q^2) = \frac{-2m}{\hbar q}\int_0^\infty dx\, x \sin\left(\frac{qx}{\hbar}\right) V(x)$$

TO MAKE THINGS CONCRETE NEED THEORETICAL MODEL OF $V(x)$

TAKE COULOMB INTERACTION $\rightarrow$ RUTHERFORD SCATTERING

$$V(x) = \frac{q_1 q_2}{x} = \frac{z Z e^2}{x}$$

z = BEAM CHARGE

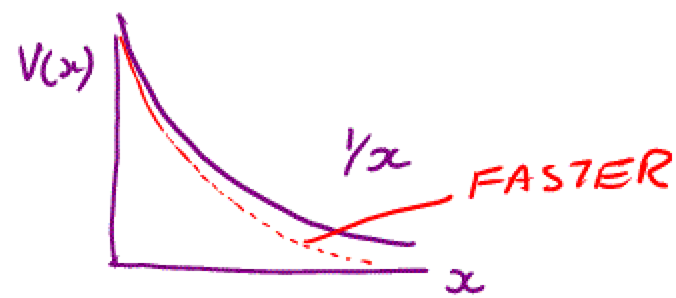
Z = TARGET CHARGE

$$f(q^2) = \frac{-2m}{\hbar q}\int_0^\infty dx \cdot x \sin\left(\frac{qx}{\hbar}\right) \cdot \frac{zZe^2}{x}$$

IS, UNFORTUNATELY, DIVERGEANT.

ASSUME NUCLEAR TARGET SHIELDED BY ATOMIC ELECTRONS

$$V(x) = \frac{zZe^2}{x} \exp\left(\frac{-x}{a}\right)$$



PARAMETER

~ DIMESIONS OF ATOM

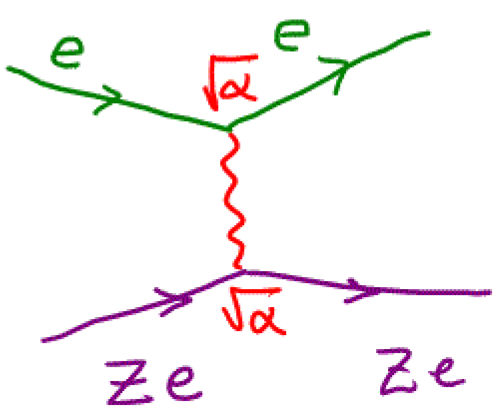
$$f(q^2) = \frac{2m\, zZe^2}{q^2 + \frac{\hbar^2}{a^2}}$$

$$f(q^2) = \frac{2m\, z Z e^2}{q^2 + \left(\frac{\hbar}{a}\right)^2}$$

WE CAN CHOOSE q^2 SUCH THAT THIS TERM NEGLIGIBLE

$$f(q^2) = \frac{2m z Z e^2}{q^2} \longrightarrow \left(\frac{d\sigma}{d\Omega}\right)_{RUTH} = \frac{4 m^2 (z Z e^2)^2}{q^4}$$

NOTE THAT FEYNMAN RULES GIVE



$$\longrightarrow \quad f(q^2) = \text{CONSTANT} \cdot \frac{z Z e^2}{q^2}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RUTH} = \frac{4m^2(zZe^2)^2}{q^4}$$

GO BACK TO LECTURE #3 $\quad q = 2p \sin\theta/2$

ASSUME NON RELATIVISTIC

$$T = \frac{1}{2} m v^2 \longrightarrow T^2 = \frac{p^4}{4m^2} = \frac{q^4}{4m^2 16 \sin^4\frac{\theta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{(zZe^2)^2}{T^2 16 \sin^4\theta/2}$$

OR

$$\frac{d\sigma}{d\Omega} = \frac{4m^2(zZe^2)^2}{p^4 \sin^4\frac{\theta}{2}}$$

RUTHERFORD DIFFERENTIAL CROSS SECTION IS:

A THEORETICAL MODEL

VALIDITY DEPENDS ON VALIDITY OF ASSUMPTIONS

- COULOMB POTENTIAL – WEAK PERTURBATION
- ONLY HEAVY TARGET — NO RECOIL
- SPIN 0 PROJECTILE & TARGET
- NO STRUCTURE, OR SPATIAL EXTENT OF TARGET OR BEAM PROJECTILE

DEVIATIONS FROM RUTHERFORD GIVE INFORMATION ON:

- DIFFERENT POTENTIAL ACTING
- NON POINT PARTICLES
- INTERNAL STRUCTURE
- SPIN OF PARTICLES

MOTT DIFFERENTIAL CROSS SECTION

ELECTRON BEAMS – RELATIVISTIC, SPIN 1/2 (E.G. SLAC LINAC) – NEED A GENERALIZATION OF RUTHERFORD – RELATIVISTIC SPIN 1/2 BEAM

$$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT} = 4\left(Ze^2\right)^2 \frac{E^2}{(qc)^4}\left(1-\underbrace{\beta^2 \sin^2\frac{\theta}{2}}\right)$$

INCOMING ELECTRON ENERGY

RELATIVISTIC EFFECT OF ELECTRON SPIN HAVING MAGNETIC INTERACTION WITH TARGET

$\rightarrow 0$ AS $\beta \rightarrow 0$; $E \rightarrow mc^2$

$$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT} \rightarrow 4\left(Ze^2\right)^2 \frac{m^2c^4}{q^4c^4} = 4\left(Ze^2\right)^2 \frac{m^2}{q^4} = \left(\frac{d\sigma}{d\Omega}\right)_{RUTH}$$

RUTHERFORD & MOTT IGNORE SPATIAL EXTENT OR INTERNAL STRUCTURE OF TARGET

HOWEVER, WE DO EXPERIMENTS TO DETERMINE EXPERIMENTALLY:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{EXPERIMENT}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{MOTT}} \cdot \left|F(q^2)\right|^2$$

MEASURE

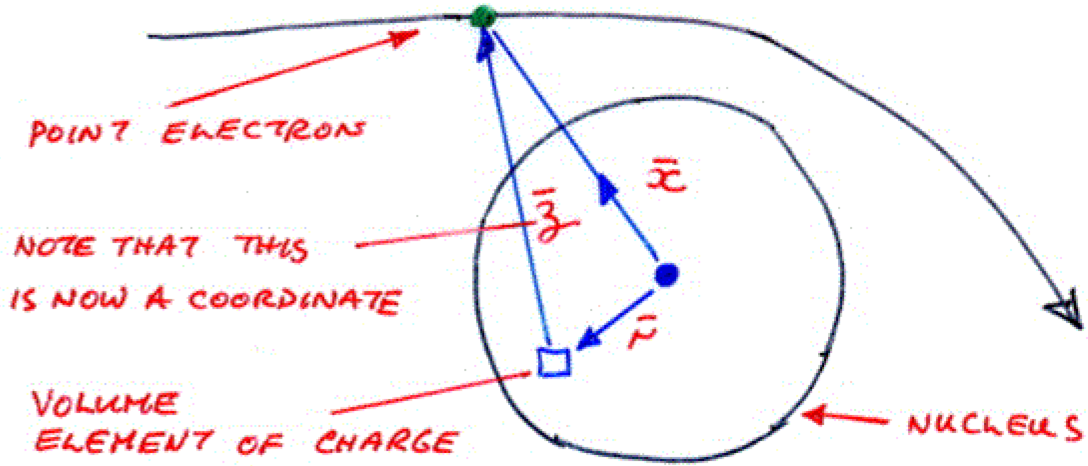
CALCULABLE POINT SCATTERING

FORM FACTOR INCORPORATES SPATIAL EXTENT

Eg. DISTRIBUTION OF ELECTRIC CHARGE

EXPERIMENT + THEORY → INFER →

RUTHERFORD SCATTERING FROM EXTENDED TARGET



CHARGE DENSITY IN NUCLEUS $\rho(r)$

IN VOLUME ELEMENT $d^3r = r^2 \sin\theta \, d\theta \, d\phi \, dr$

CHARGE IS $Ze\rho(r)\,d^3r$ CONTRIBUTION OF THIS ELEMENT TO COULOMB POTENTIAL AT ELECTRON

$$dV(x) = \frac{-Ze^2}{z} \underbrace{\exp\left(\frac{-z}{a}\right)}_{\text{SHIELDING}} \rho(r)\, dr$$

COULOMB

CHARGE DENSITY

$$dV(x) = \frac{-Ze^2}{z} \exp\left(\frac{-z}{a}\right) \rho(r)\, dr$$

POTENTIAL AT x DUE TO WHOLE NUCLEUS

$$V(x) = -Ze^2 \int d^3r \, \frac{\rho(r)}{z} \exp\left(\frac{-z}{a}\right)$$

PUT THIS POTENTIAL INTO BORN APPROXIMATION

$$f(q^2) = \frac{-m}{2\pi\hbar^2} \int V(x)\, e^{-i\bar{q}\cdot x/\hbar}\, d^3\bar{x}$$

NEW COORDINATE SYSTEM $\bar{x} = \bar{r} + \bar{z}$

$$f(q^2) = \frac{mZe^2}{2\pi\hbar^2} \int d^3r\, e^{i\bar{q}\cdot\bar{r}/\hbar} \rho(r) \int d^3z\, \frac{e^{-z/a}}{z}\, e^{i\bar{q}\cdot\bar{z}/\hbar}$$

$$f(q^2) = \frac{m Z e^2}{2\pi\hbar^2} \underbrace{\int d^3r\, e^{i\vec{q}\cdot\vec{r}} \rho(r)} \underbrace{\int d^3\xi\, \frac{e^{-\xi/a}}{\xi} e^{i\vec{q}\cdot\vec{\xi}/\hbar}}$$

INTEGRAL OVER VOLUME OF NUCLEUS

FORM FACTOR OF NUCLEUS

$$F(q^2) = \int d^3r\, e^{i\vec{q}\cdot\vec{r}/\hbar} \rho(r)$$

FOURIER TRANSFORM OF SPATIAL CHARGE DENSITY

CHARGE DENSITY IN MOMENTUM SPACE

INTEGRAL OVER ELECTRON ORBIT OF COULOMB INTERACTIONS

$$= \frac{4\pi\hbar^2}{q^2 + \left(\frac{\hbar}{a}\right)^2} \rightarrow \frac{4\pi\hbar^2}{q^2}$$

$$\sim \frac{1}{q^2} \rightarrow \text{MOMENTUM TRANSFER}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \left|f(q^2)\right|^2 = \left|F(q^2)\right|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{POINT TARGET}}$$

USE DIFFERENT BEAMS TO PROBE DIFFERENT ASPECTS OF SUBATOMIC MATTER

SIMPLEST IF BEAM IS A POINT PARTICLE

ELECTRONS — PROBE ELECTRIC CHARGE DISTRIBUTION

NEUTRINOS — PROBE DISTRIBUTION OF WEAK CHARGE

HADRONS (BOUND STATES OF QUARKS) — PROBE DISTRIBUTION OF STRONGLY INTERACTING MATTER

NOT POINT LIKE BEAM
SO HAVE CONVOLUTION OF BEAM AND TARGET FORM FACTORS

FROM FORM FACTOR GET ANOTHER INSIGHT INTO WHAT BEAM MOMENTUM IS NEEDED TO RESOLVE STRUCTURE IN TARGET

$$F(q^2) = \int d^3r\, e^{i\bar{q}\cdot\bar{r}/\hbar} \rho(r)$$

IF ~ 1 THEN $F(q^2) \doteq$ CHARGE ON NUCLEUS

POINT SCATTERING

FOR $F(q^2)$ TO BE DIFFERENT FROM POINT SCATTERING

$$e^{i\bar{q}\cdot\bar{r}/\hbar} < 1 \rightarrow \frac{\bar{q}\cdot\bar{r}}{\hbar} \sim 1$$

SO $\bar{q} \sim \hbar/r \rightarrow q > 20\ \mathrm{MeV}/c$

$197\ \frac{\mathrm{MeV}}{c}\cdot \mathrm{fm}$

10 fm FOR NUCLEAR RADIUS

EXPERIMENTAL DETERMINATION OF TARGET STRUCTURE

MEASURE FORM FACTOR $\left(\frac{d\sigma}{d\Omega}\right)_{EXPT} = |F(q^2)|^2 \frac{d\sigma}{d\Omega}$ POINT THEORY

DO A FOURIER TRANSFORM TO GET CHARGE DENSITY IN REAL SPACE

$$\rho(r) = \frac{1}{(2\pi)^3} \int d^3q \; F(q^2)\, e^{-i\vec{q}\cdot\vec{r}/\hbar}$$

FORM FACTOR

HAVE TO MEASURE OVER ALL q^2 — DIFFICULT.

USUALLY HYPOTHESIZE FORM OF $\rho(r)$

PUT INTO $F(q^2) = \int d^3r \, e^{i\vec{q}\cdot\vec{r}/\hbar} \rho(r)$ AND

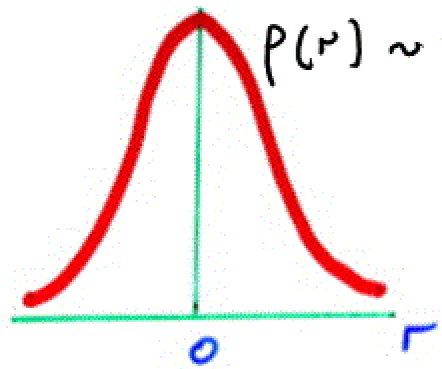
DETERMINE WHICH $\rho(r)$ GIVES BEST FIT TO THE EXPERIMENTAL $F(q^2)$

ELECTRIC CHARGE DISTRIBUTIONS IN NUCLEI

TWO SIMPLE MODELS

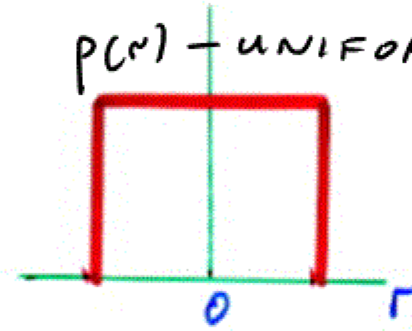
GAUSSIAN

$$F(q^2) = \exp\left[\frac{-q^2 b^2}{4\pi^2}\right]$$



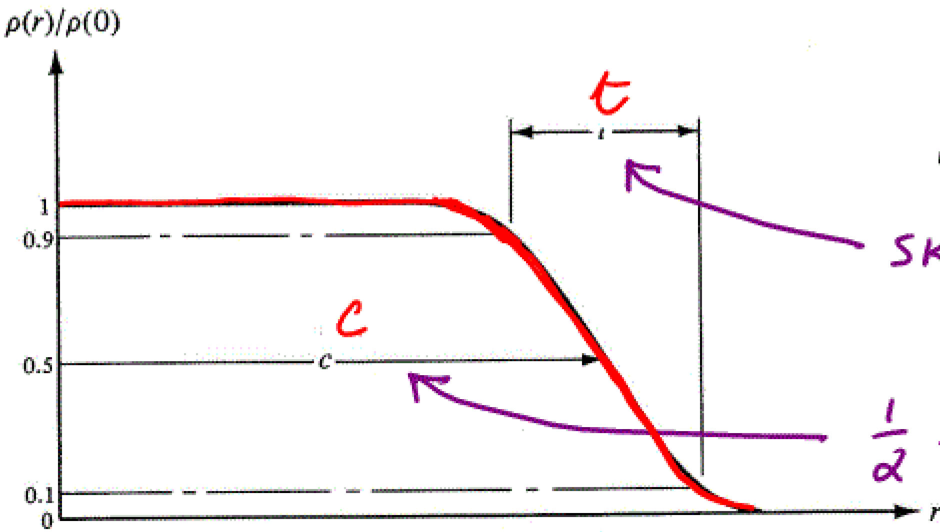
$$\rho(r) \sim e^{-(r/b)^2}$$

$\rho(r)$ – UNIFORM



$F(q^2)$ COMPLEX

STANDARD NUCLEAR MODEL



$$\rho(r) = \frac{1}{1 + e^{(r-c)a}}$$

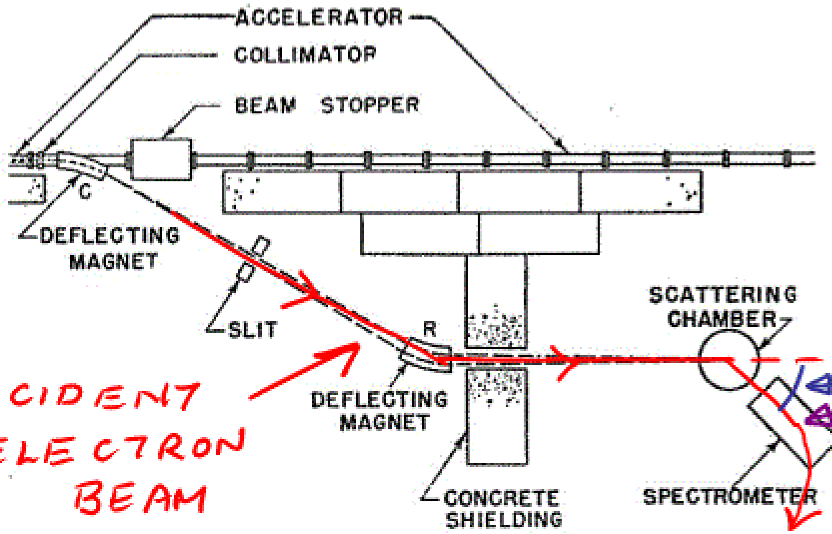
SKIN THICKNESS

$$t = (4 \ln 3)\, a$$

$\frac{1}{2}$ DENSITY RADIUS OF NUCLEUS

HOFSTADTER 1953

STANFORD 550 MeV LINAC



INCIDENT ELECTRON BEAM

SCATTERING ANGLE

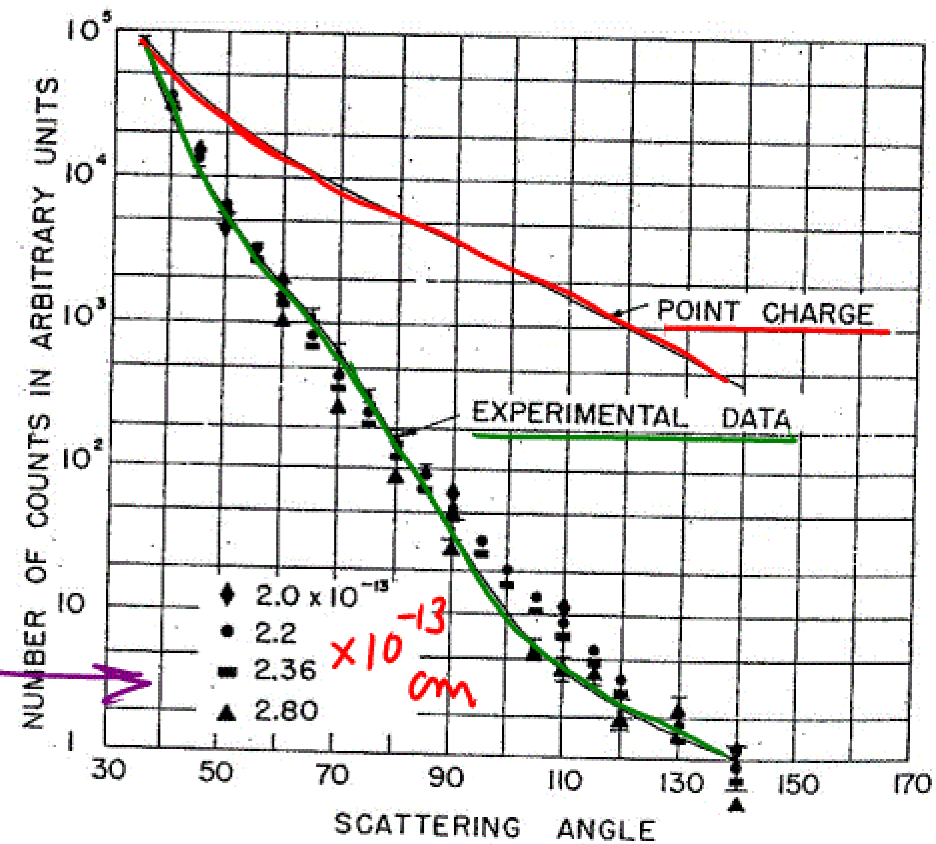
SCATTERED ELECTRON BEAM

NUCLEAR SIZE

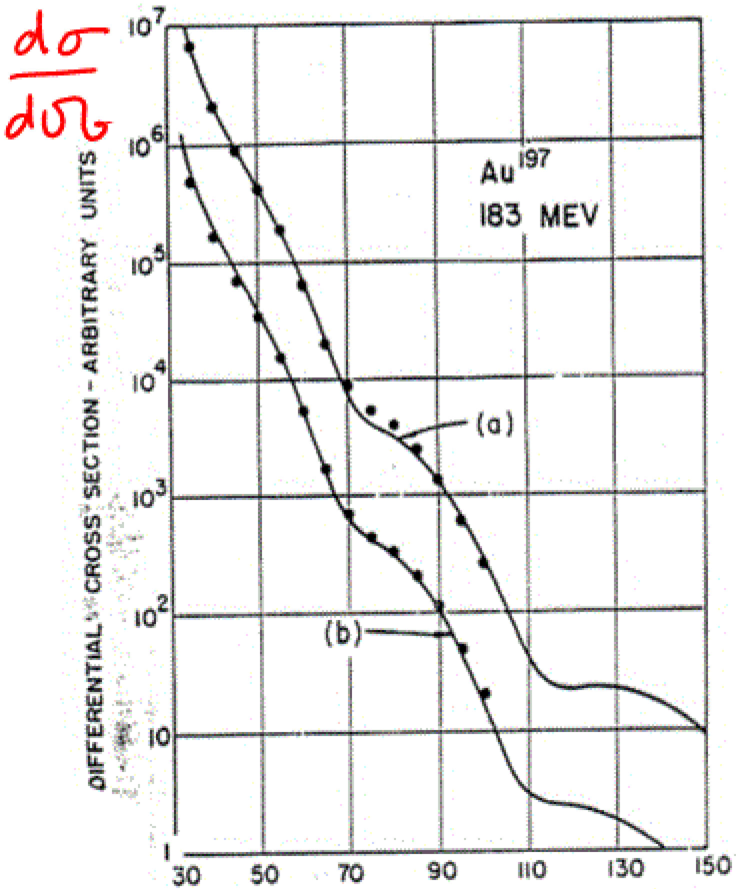
$eAu \to eAu$

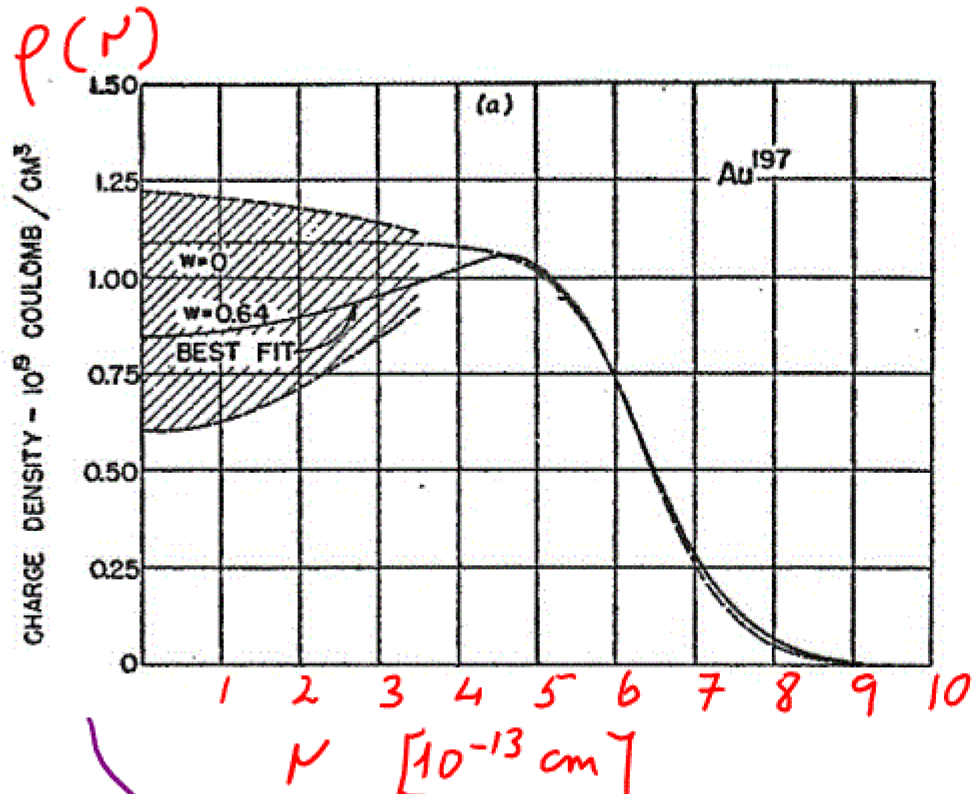
NUCLEUS NOT A POINT PARTICLE

NUCLEAR RADIUS HYPOTHESIZED IN GAUSSIAN FORM FACTOR



DETERMINATION OF CHARGE DISTRIBUTION IN NUCLEI

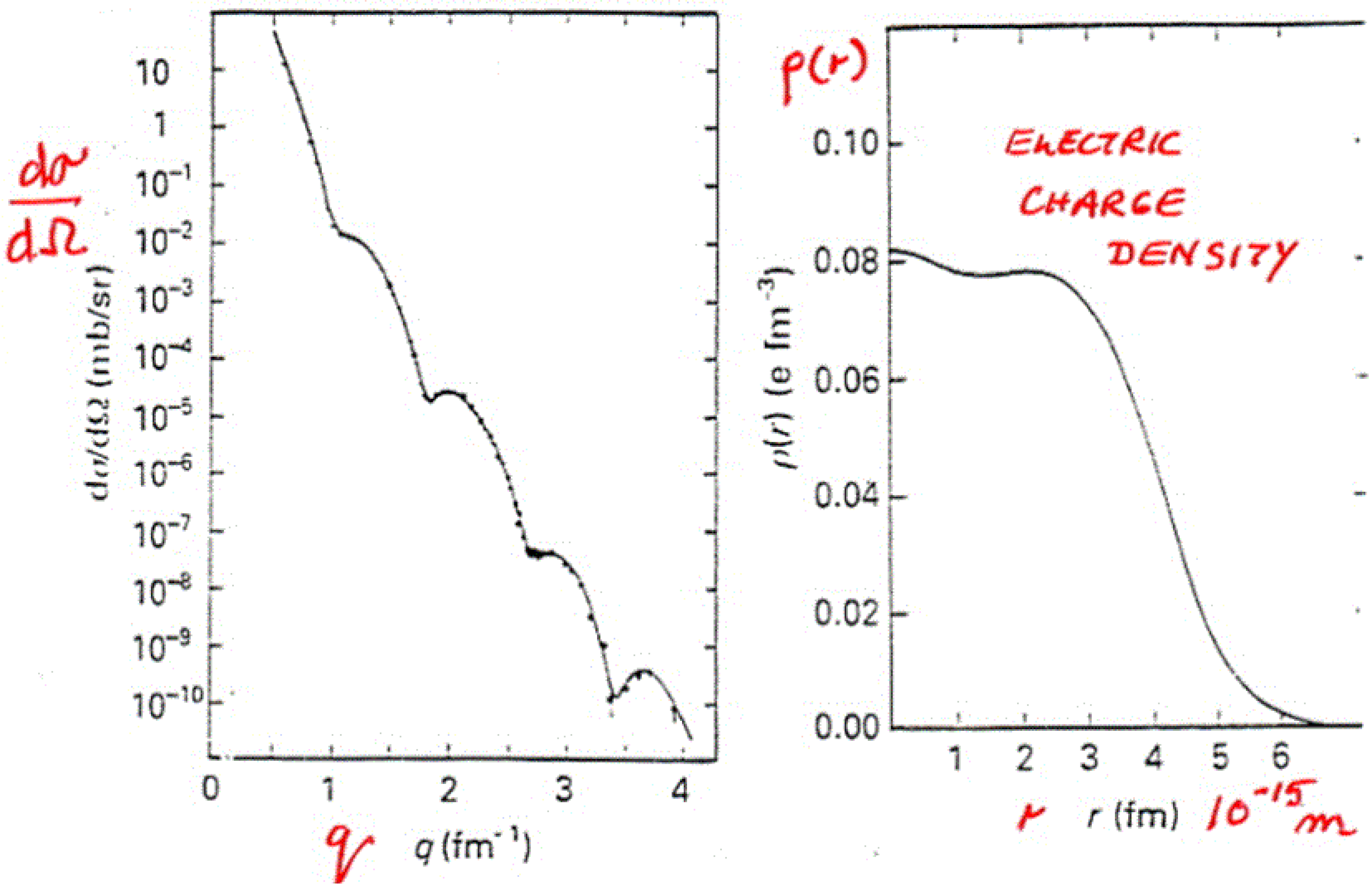




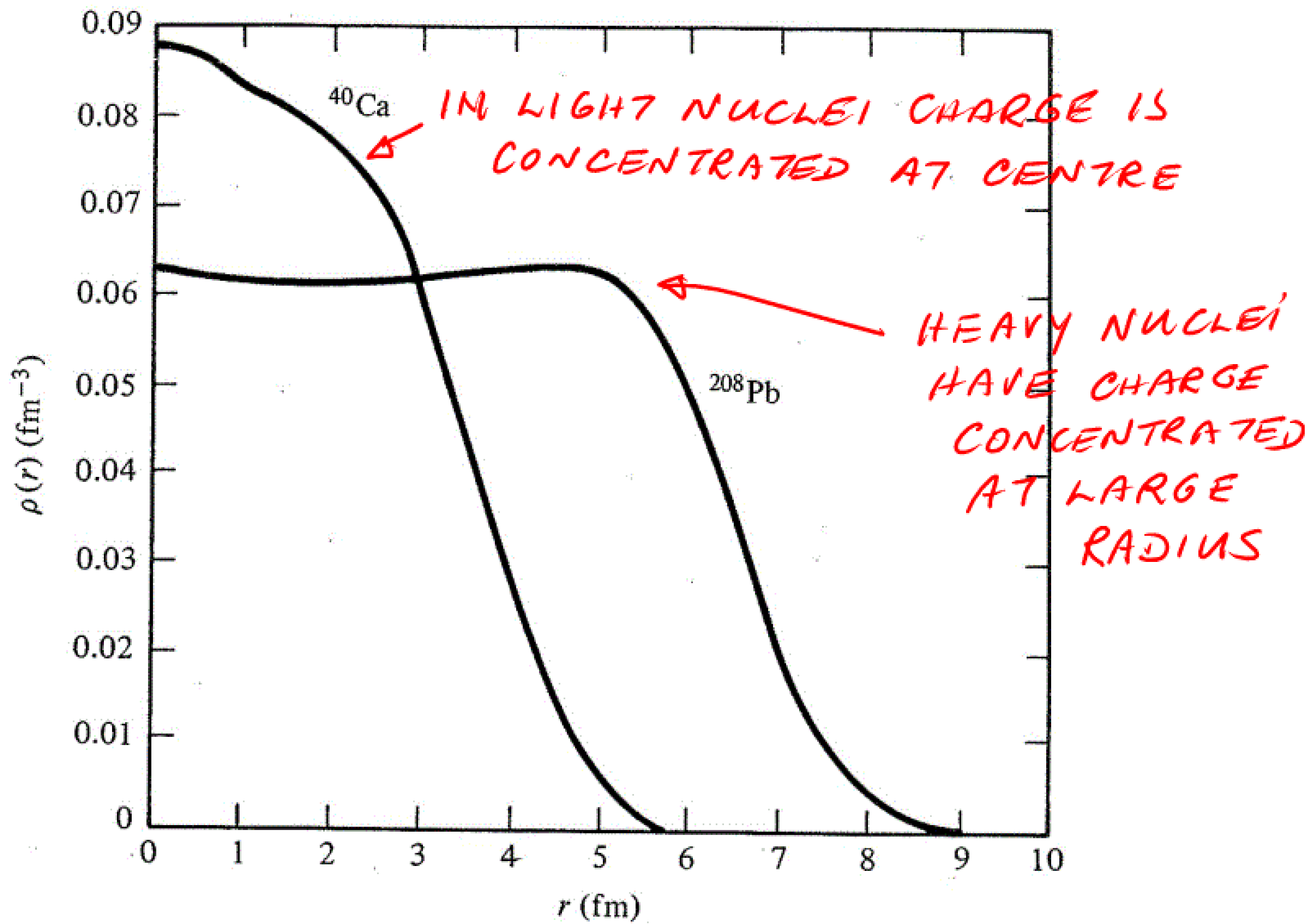
$$\rho(r) = \frac{1}{(2\pi)^3} \int d^3q \, F(q^2) \, e^{-i\vec{q}\cdot\vec{r}/\hbar}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{MOTT} \\ \text{(POINT)}}} \left|F(q^2)\right|^2$$

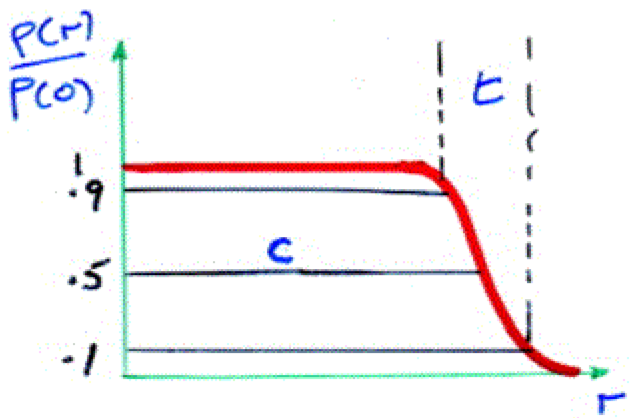
450 MeV ELECTRONS ON $^{58}_{28}Ni$



COMPARISON OF LIGHT & HEAVY NUCLEI



ELECTRIC CHARGE DISTRIBUTION IN NUCLEI



$$\rho(r) = \frac{\rho_0}{1 + e^{(r-c)/a}}$$

$$t = (4 \ln 3)\, a$$

REPRESENT AN OBJECT WITH DIFFUSE BOUNDARY BY

ROOT MEAN SQUARE CHARGE RADIUS $\langle r^2 \rangle^{1/2}$

$$\langle r^2 \rangle = \int d^3r \, r^2 \rho(r)$$

THESE EXPERIMENTS SHOW

$$\langle r^2 \rangle^{\frac{1}{2}} = r_0 \; A^{1/3}$$

0.94 fm

MASS NUMBER = SUM OF PROTONS AND NEUTRONS

$$\langle r^2 \rangle^{\frac{1}{2}} = r_0 \, A^{1/3}$$

- VOLUME $\propto$ (RADIUS)^3 $\rightarrow$ NUCLEAR VOLUME $\propto (A^{1/3})^3 \propto A$
- NUCLEAR MASS CLEARLY $\propto$ A

$\therefore$ DENSITY OF NUCLEAR MATTER CONSTANT FOR ALL NUCLEI (EXCEPT LIGHTEST)

- ELECTRON EXPERIMENTS

$$C = 1.18\,A^{-1/3} - 0.48 \text{ fm}$$
$$t = 2.4 \text{ fm}$$

$\rightarrow$ $\rho_{\text{NUCLEUS}} \approx 0.17$ NUCLEON / fm^3