CONTINUOUS TRANSFORMATIONS

- MATHEMATICALLY USUALLY EASIER TO CONSIDER INFINITESIMAL TRANSFORMATIONS
- EXAMPLE → BUILD FINITE ROTATION FROM MANY INFINITESIMAL ROTATIONS

FINITE CONTINUOUS TRANSFORMATION $= \sum$ INFINITESIMAL CONTINUOUS TRANSFORMATIONS

STUDY PROPERTIES

UNITARY TRANSFORMATIONS & SYMMETRIES

A MATHEMATICAL REPRESENTATION OF A UNITARY TRANSFORMATION ; $U^{+}U = 1$

REAL PARAMETER

$$U = e^{i\varepsilon F}$$

GENERATOR OF TRANSFORMATION

IF U IS A UNITARY OPERATOR

F GENERATES A GROUP OF TRANSFORMATIONS

GROUP MEMBER

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \in G$$

$$g \cdot I = g = I \cdot g$$

$$g \cdot g^{-1} = I = g^{-1} \cdot g$$

IF g_i AND $g_j \in G \rightarrow g_i \cdot g_j \in G$

GROUPS

ABSTRACT ENTITIES → PHYSICAL SYMMETRIES

SO(3) → GROUP OF ROTATIONS ABOUT X, Y, Z → CONSERVATION OF ANGULAR MOMENTUM

U(1) → GROUP OF 1-d ROTATIONS $e^{i\alpha}$

SU(2) → GROUP OF ALL 2×2 MATRICES WITH UNIT DETERMINANT

$\begin{pmatrix} e \\ \nu \end{pmatrix}$ SYMMETRY OF WEAK FORCE

SU(3) → 3×3

$\begin{matrix} r \\ g \\ b \end{matrix}$ SYMMETRY OF COLOUR FORCE

$U = e^{i\varepsilon F}$ ACTION ON WAVE FUNCTION

$$\psi \to U\psi = e^{i\varepsilon F}\psi = \left(1 + i\varepsilon F + \frac{(i\varepsilon F)^2}{2!} + \cdots\cdots\right)\psi$$

GENERALLY $e^{i\varepsilon F} \neq e^{i\varepsilon F^+}$ SINCE $F \neq F^+$

U IS NOT GENERALLY HERMITIAN $\to$ NOT OBSERVABLE

IF U IS UNITARY $\to U^+U = 1$

$$e^{-i\varepsilon F^+} e^{i\varepsilon F} = 1$$

$$e^{i\varepsilon(F - F^+)} = 1$$

$F = F^+ \longrightarrow$ SO F, THE GENERATOR OF U IS HERMITIAN & OBSERVABLE

$\mathcal{U}$ UNITARY $\longrightarrow$ GENERATOR F OBSERVABLE

ASSUME $\mathcal{U}$ IS ALSO SYMMETRY OPERATOR $\rightarrow [H, \mathcal{U}] = 0$

$$\mathcal{U} = e^{i\varepsilon F} \rightarrow \mathcal{U} = 1 + i\varepsilon F \quad \text{FOR } \varepsilon F \ll 1$$

$$\begin{aligned}
[H, \mathcal{U}] &= H\mathcal{U} - \mathcal{U}H \\
&= H(1 + i\varepsilon F) - (1 + i\varepsilon F)H \\
&= H + i\varepsilon HF - H - i\varepsilon FH \\
&= i\varepsilon HF - i\varepsilon FH \\
&= i\varepsilon [H, F]
\end{aligned}$$

IF $[H, \mathcal{U}] = 0 \longrightarrow [H, F] = 0$

$\mathcal{U}$ IS A SYMMETRY OF THE HAMILTONIAN $\Rightarrow$ GENERATOR F IS A CONSERVED OBSERVABLE

SYMMETRY	CONSERVED QUANTITY
SPACE TRANSLATION	MOMENTUM
ROTATION	ANGULAR MOMENTUM
TIME	ENERGY
ROTATION IN ISOSPACE	ISOSPIN
?	LEPTON FLAVOR
?	QUARK FLAVOR
?	BARYON NUMBER
GLOBAL SU(3) GAUGE	COLOUR CHARGE
GLOBAL SU(2) GAUGE	WEAK CHARGE
GLOBAL U(1) GAUGE	ELECTRIC CHARGE
LOCAL U(1) GAUGE	EM FORCE
LOCAL SU(2) GAUGE	WEAK FORCE
LOCAL SU(3) GAUGE	COLOUR FORCE

GENERAL HAMILTONIAN FOR PARTICLES

LAGRANGIAN DENSITY IN FIELD THEORY

$$\mathcal{H} = \mathcal{H}_{COLOR} + \mathcal{H}_{EM} + \mathcal{H}_{WEAK} + \mathcal{H}_{GRAVITY}$$

EACH $\mathcal{H}$ → DIFFERENT SYMMETRIES → DIFFERENT CONSERVATION LAWS

$$|\text{INITIAL STATE}\rangle \xrightarrow{\mathcal{H}} |\text{FINAL STATE}\rangle$$

DETERMINES WHICH QUANTUM NUMBERS CONSERVED

↓

DETERMINES WHICH $\mathcal{H}$ ACTING

MEASURING CONSERVED QUANTUM NUMBER → DETERMINE SYMMETRIES OF HAMILTONIAN → DEFINE FORM OF HAMILTONIAN

EG $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

$\mu^- \not\to e^- \gamma$

} LEPTON NUMBER CONSERVATION IS A SYMMETRY OF $\mathcal{H}_{wk}$

$n \to p\, e^- \bar{\nu}_e$

$n \not\to \pi^+ e^- \bar{\nu}_e$

} DOES CONSERVATION OF BARYON NUMBER IMPLY

$$[\mathcal{H}_{STRONG}, B] = 0$$

DEPENDS ON WHETHER WE SEE → EXPERIMENT!

BARYON # OPERATOR

CONSERVATION OF ELECTRIC CHARGE

$$e^- \not\to \nu \gamma \quad (\tau > 2 \times 10^{21} \text{ years})$$

CONSERVATION OF ELECTRIC CHARGE → SYMMETRY

$$i\hbar \frac{d\psi}{dt} = H\psi$$

Q → AN OPERATOR CORRESPONDING TO "ELECTRIC" CHARGE

$\langle Q \rangle$ CONSERVED IF $[H, Q] = 0$

ψ CAN BE AN EIGNFUNCTION OF BOTH ENERGY AND CHARGE

$$Q\psi = q\psi$$

PHYSICS IS INVARIANT UNDER PHASE TRANSFORMATION

$$\psi \to e^{i\varepsilon Q}\psi$$

GLOBAL GAUGE (PHASE) TRANSFORMATION

INDEPENDANT OF x, t

$$\psi' = e^{i\varepsilon Q}\psi$$

$$\psi\psi^* = (\psi')^*\psi'$$

PROBABILITIES UNCHANGED

GAUGE INVARIANCE MEANS THAT

$$i\hbar\frac{d\psi'}{dt} = \mathcal{H}\psi'$$

SAME DYNAMICS

JUST AS!

$$i\hbar\frac{d\psi}{dt} = \mathcal{H}\psi$$

$$i\hbar \frac{d\psi'}{dt} = \mathcal{H}\psi'$$

REWRITE IN TERMS OF UNTRANSFORMED WAVE FUNCTION

$$i\hbar \frac{d}{dt} e^{i\varepsilon Q}\psi = \mathcal{H} e^{i\varepsilon Q}\psi$$

$$i\hbar \frac{d}{dt}\left(e^{-i\varepsilon Q^{+}} e^{i\varepsilon Q}\right)\psi = e^{-i\varepsilon Q^{+}}\mathcal{H} e^{i\varepsilon Q}$$

← MULT

ASSUME Q OBSERVABLE → HERMITIAN

$Q^{+} = Q$

$$i\hbar \frac{d\psi}{dt} = \underbrace{e^{-i\varepsilon Q}\mathcal{H} e^{i\varepsilon Q}}_{}\psi$$

cf $\quad i\hbar \dfrac{d\psi}{dt} = \mathcal{H}\psi$

$$i\hbar\, d\psi/dt = e^{-i\varepsilon Q} \mathcal{H}\, e^{i\varepsilon Q} \cdot \psi$$

SAME DYNAMICS

COMPARE TO: $i\hbar\, d\psi/dt = \mathcal{H}\,\psi$

THEN $e^{-i\varepsilon Q} \mathcal{H} e^{i\varepsilon Q} = \mathcal{H}$

EXPAND EXPONETIAL — INFINITESIMAL GAUGE TRANSFORM $\rightarrow \varepsilon Q \rightarrow 0$

$$(1 - i\varepsilon Q)\mathcal{H}(1 + i\varepsilon Q) = \mathcal{H}$$

$\rightarrow 0$

$$\mathcal{H} + i\varepsilon \mathcal{H} Q - i\varepsilon Q \mathcal{H} - \cancel{(i\varepsilon)^2 Q \mathcal{H} Q} = \mathcal{H}$$

$$[\mathcal{H} Q - Q \mathcal{H}] = \mathcal{H} - \mathcal{H} = 0$$

$\rightarrow [\mathcal{H}, Q] = 0$

Q CONSERVED

$e^{i\varepsilon Q}$ IS A SYMMETRY

$\Downarrow$

Q IS A CONSERVED QUANTITY.

- GLOBAL GAUGE SYMMETRY $\longrightarrow$ CONSERVED CHARGE

- PHYSICS MUST BE INVARIANT UNDER ARBITRARY PHASE CHANGES AT DIFFERENT POINTS IN SPACE-TIME

CHANGES CANNOT BE TRANSMITTED INSTANTANEOUSLY $\longrightarrow$ SPECIAL RELATIVITY

$$\psi \to \psi e^{i\varepsilon(\bar{x},t)\,Q}$$

LOCAL GAUGE INVARIANCE $\longrightarrow$ FORCES

A TASTE OF LOCAL GAUGE SYMMETRY

LOCAL GAUGE INVARIANCE

$$\psi \to \psi e^{i\alpha(\bar{x},t)}$$

TOY MODEL

ARBITRARY CHANGE AT EVERY SPACE-TIME POINT

ILLUSTRATION → JUST SPATIAL DEPENDENCE

$$H\psi(\bar{x}) = E\psi(\bar{x})$$

HAMILTONIAN $\quad E_{TOTAL} = \text{KINETIC} + \text{POTENTIAL}$

$$= \frac{p^2}{2m} + V(\bar{x})$$

SO

$$H = \frac{-\hbar^2}{2m}\bar{\nabla}^2 + V(\bar{x})$$

KINETIC

POTENTIAL

IF $\psi \to \psi e^{i\alpha(\bar{x})}$ DO WE STILL HAVE SAME SCHRÖDINGER? → INVARIANCE

SINCE $\bar{\nabla}\left[e^{i\alpha(\bar{x})}\psi(\bar{x})\right] \neq e^{i\alpha(\bar{x})}\bar{\nabla}\psi(\bar{x})$

IN $\left[\frac{-\hbar^2}{2m}\bar{\nabla}^2 + V(\bar{x})\right]\psi(\bar{x}) = E(\bar{x})\,\psi(\bar{x})$

THE RIGHT & LEFT HAND SIDES BEHAVE DIFFERENTLY

→ SO DO NOT HAVE INVARIANCE UNDER

$$\psi \to e^{i\alpha(\bar{x})}\psi$$

BUT MUST BE SINCE

$$\psi\psi^* = e^{i\alpha(\bar{x})}\psi\left(e^{i\alpha(\bar{x})}\psi\right)^*$$

HOW TO ENSURE LOCAL GAUGE INVARIANCE?

POSTULATE THAT HAMILTONIAN IS INCOMPLETE

$$\left[-\frac{\hbar}{2m}\bar{\nabla}^2 + V(\bar{x})\right]\psi(\bar{x}) = E\,\psi(\bar{x})$$

A NUMBER

UNDER $\psi \to e^{i\alpha(\bar{x})}\psi$ RHS $E\,\psi(\bar{x}) \to E\,e^{i\alpha(\bar{x})}\psi(\bar{x})$

$$= e^{i\alpha(\bar{x})}E\,\psi(\bar{x})$$

FOR INVARIANCE NEED LHS:

$$\left[-\frac{\hbar^2}{2m}\bar{\nabla}^2 + V(x)\right]e^{i\alpha(\bar{x})}\psi(\bar{x}) = e^{i\alpha(\bar{x})}\left[-\frac{\hbar^2}{2m}\bar{\nabla}^2 + V(x)\right]\psi(\bar{x})$$

BUT

$$\bar{\nabla}(\bar{x})\left[e^{i\alpha(\bar{x})}\psi(\bar{x})\right] = e^{i\alpha(\bar{x})}\left[i\bar{\nabla}\alpha(\bar{x})\,\psi(\bar{x}) + \bar{\nabla}\psi(\bar{x})\right]$$

$$\neq e^{i\alpha(\bar{x})}\psi(\bar{x})$$

CANNOT BE TRUE $\longrightarrow$ NO LOCAL GAUGE INVARIANCE

INTRODUCE A NEW VECTOR $A(\bar{x})$

DEFINE COVARIANT DIFFERENTIAL

$\bar{\nabla} \to \bar{\nabla} - i\bar{A}(x)$ AND $\bar{A}(\bar{x}) \to \bar{A}(\bar{x}) + \bar{\nabla}\alpha(\bar{x})$

UNDER LOCAL GAUGE TRANSFORMATION $\psi(\bar{x}) \to e^{i\alpha(\bar{x})}\psi(\bar{x})$

WE NOW HAVE A NEW SCHRÖDINGER

$$\left\{\frac{-\hbar^2}{2m}\left(\bar{\nabla} - i\bar{A}(\bar{x})\right)^2 + V(\bar{x})\right\}\psi(\bar{x}) = E\psi(\bar{x})$$

UNDER LOCAL GAUGE TRANSFORM

$$\left\{\frac{-\hbar^2}{2m}\left(\underbrace{\bar{\nabla} - i\left(A(\bar{x}) + \bar{\nabla}\alpha(\bar{x})\right)}\right)^2 + V(\bar{x})\right\}e^{i\alpha(\bar{x})}\psi(\bar{x}) = e^{i\alpha(\bar{x})}E\psi(\bar{x})$$

LOOK AT THIS PART

$$(\bar{\nabla} - i\bar{A}(\bar{x}) - i\bar{\nabla}\alpha(\bar{x}))\, e^{i\alpha(\bar{x})}\psi(\bar{x})$$

$$= \underbrace{\bar{\nabla}\left[e^{i\alpha(\bar{x})}\psi(\bar{x})\right]} - iA(\bar{x})e^{i\alpha(\bar{x})}\psi - i\bar{\nabla}\alpha(\bar{x})e^{i\alpha(\bar{x})}\psi(\bar{x})$$

$$= \cancel{i\bar{\nabla}\alpha(\bar{x})e^{i\alpha(\bar{x})}}\psi + e^{i\alpha(\bar{x})}\bar{\nabla}\psi(\bar{x}) - iA(\bar{x})e^{i\alpha(\bar{x})}$$

$$- \cancel{i\bar{\nabla}\alpha(\bar{x})e^{i\alpha(\bar{x})}}\psi(\bar{x})$$

$$= e^{i\alpha(\bar{x})}(\bar{\nabla} - i\bar{A}(\bar{x}))\psi(\bar{x})$$

$$(\bar{\nabla} - iA(\bar{x}))\psi(\bar{x}) \xrightarrow{\text{GAUGE}} e^{i\alpha(\bar{x})}(\bar{\nabla} - iA(\bar{x}))\psi(\bar{x})$$

CAN JUST PULL THRU

$$(\bar{\nabla} - iA(\bar{x}))^2\psi(\bar{x}) \xrightarrow{\text{GAUGE}} e^{i\alpha(\bar{x})}(\bar{\nabla} - iA(\bar{x}))\psi(\bar{x})$$

INTRODUCING $A(\bar{x})$ AND ITS TRANSFORMATION

$$A(\bar{x}) \to A(\bar{x}) + \bar{\nabla}\alpha(\bar{x})$$

FORCES GAUGE INVARIANCE

$$\left\{ -\frac{\hbar^2}{2m} \left(\bar{\nabla} - i\bar{A}(\bar{x}) \right)^2 + V(\bar{x}) \right\} \psi(\bar{x}) = E\,\psi(\bar{x})$$

ORIGINAL KINETIC ENERGY

NEW KINETIC ENERGY DESCRIBES MOTION OF NEW PARTICLE

"VECTOR POTENTIAL" $A(\bar{x})$

LOCAL GAUGE INVARIANCE → E.M. FORCE → γ

REQUIRED SYMMETRY $\psi \to \underbrace{e^{i\alpha(\bar{x})}}_{\text{GROUP } U(1)} \psi$

$U(1)$ → GAUGE SYMMETRY OF EM FORCE

LET'S LOOK AT PUTTING TIME IN

$$\psi(\bar{x}) \longrightarrow \psi(\bar{x}, t)$$

$$\alpha(\bar{x}) \longrightarrow \alpha(\bar{x}, t)$$

WE CAN ANTICIPATE THAT THE TIME VARIATION OF α WILL HAVE TO BE CANCELLED JUST AS SPACE VARIATION SO WRITE.

$$\left\{ -\frac{\hbar^2}{2m} \left(\bar{\nabla} - i\bar{A}(\bar{x}, t) \right)^2 + V(\bar{x}) \right\} \psi(\bar{x}, t)$$

$$- \hbar \phi(\bar{x}, t) \psi(\bar{x}, t) = E \psi(\bar{x})$$

AND DEFINE OUR GAUGE TRANSFORMATION TO MAKE $\phi(\bar{x}, t)$ CANCEL TIME VARIATION

$$\left(\bar{\nabla} - iA(\bar{x},t)\right)^2 + V(x)\Big\}\psi(\bar{x},t) - \hbar\,\phi(\bar{x},t)\,\psi(\bar{x},t) = E\psi$$

IF $\psi(\bar{x},t) \rightarrow e^{i\alpha(\bar{x},t)}\,\psi(\bar{x},t)$

THEN $\phi(\bar{x},t) \rightarrow \phi(\bar{x},t) + d\alpha/dt$

$A(\bar{x},t) \rightarrow A(\bar{x},t) + \bar{\nabla}\alpha(\bar{x},t)$

THE $\hbar\phi\psi$ TERM GIVES AN EXTRA $-\hbar\frac{d\alpha}{dt}\psi$ ON LHS

$\rightarrow$ CANCELLATION

$\rightarrow$ LOCAL GAUGE INVARIANCE

$$A_\mu = (\phi, \bar{A})$$

$$\left\{ -\frac{\hbar^2}{2m}\left(\bar{\nabla} - i\bar{A}(\bar{x},t)\right)^2 + V(x) \right\} \psi(\bar{x},t) - \hbar\,\phi(\bar{x},t)\,\psi(\bar{x},t) = E\psi$$

CLEARY A_μ IS A 4-VECTOR $A(t, x, y, z)$

x, y, z COMPONENTS OF A $\rightarrow$ SPIN 1

$(2J+1) = 3$

OUR THEORY HAS $A\psi$ TERMS

NO AA TERMS

IF WE SAY ψ DESCRIBES AN ELECTRON

WE HAVE (STILL A TOY) THEORY

ELECTRON

SPIN 1 PARTICLE INTERACTS WITH ELECTRON

SPIN 1 PARTICLE DOES NOT INTERACT WITH ITSELF

"A $\rightarrow$ EM 4-POTENTIAL"

THERE ARE ALSO GAUGE INVARIANT

$$\frac{dA_x}{dy} - \frac{dA_y}{dx}$$

WHICH CAN BE REARANGED IN TO A WAVE EQUATION

$$\nabla^2 A = \frac{1}{c^2}\frac{d^2 A}{dt^2}$$

← WAVE EQUATION FOR A MASSLESS PARTICLE

GAUGE THEORIES

GAUGE THEORIES HAVE BEEN BUILT FOR

$$\left.\begin{array}{l}\text{COLOUR FORCE} - SU(3)\\ \text{WEAK FORCE} - SU(2)\end{array}\right\}$$

NON-ABELIAN (GENERATORS DO NOT COMMUTE)

EM FORCE $- U(1)$

ABELIAN (GENERATORS COMMUTE)

GENERALLY GAUGE INVARIANCE LEADS TO MASSLESS GAUGE BOSONS

$W^{\pm}$ Z^{0} ?? $\rightarrow$ HIGGS MECHANISM