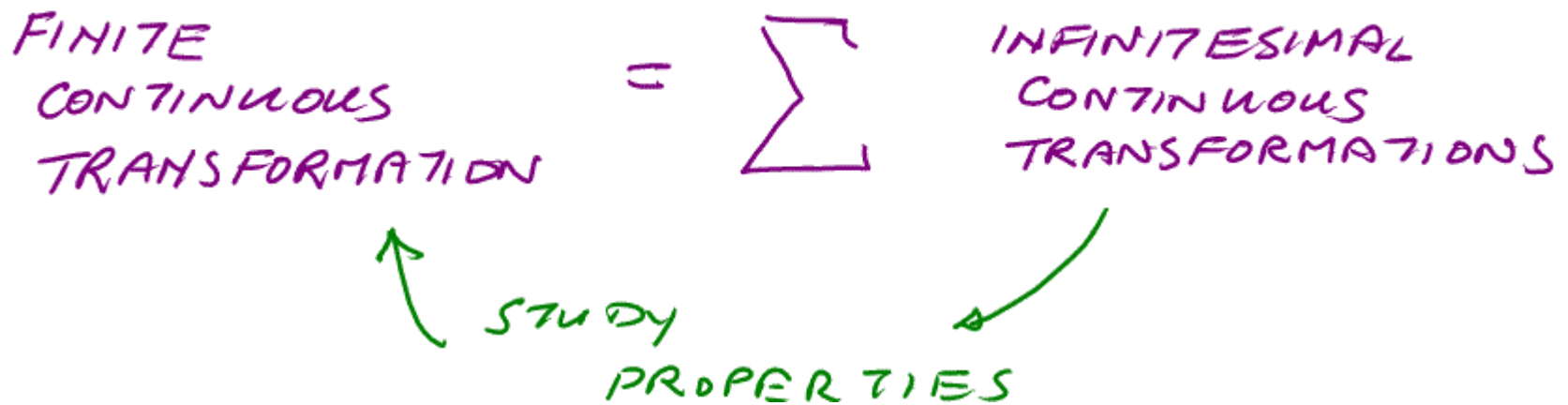


CONTINUOUS TRANSFORMATIONS

- MATHEMATICALLY USUALLY EASIER TO CONSIDER INFINITESIMAL TRANSFORMATIONS
- EXAMPLE → BUILD FINITE ROTATION FROM MANY INFINITESIMAL ROTATIONS



UNITARY TRANSFORMATIONS & SYMMETRIES

A MATHEMATICAL REPRESENTATION OF A
UNITARY TRANSFORMATION; $U^\dagger U = 1$

REAL PARAMETER

$$U = e^{i\epsilon F}$$

GENERATOR OF
TRANSFORMATION

IF U IS A UNITARY OPERATOR
 F GENERATES A GROUP OF TRANSFORMATIONS

GROUP MEMBER \rightarrow

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \in G$$

$$g \cdot I = g = I \cdot g$$

$$g \cdot g^{-1} = I = g^{-1} \cdot g$$

$$\text{IF } g_i \text{ AND } g_j \in G \rightarrow g_i \cdot g_j \in G$$

GROUPS

ABSTRACT ENTITIES \rightarrow PHYSICAL SYMMETRIES

$SO(3)$ \rightarrow GROUP OF ROTATIONS ABOUT
 x, y, z \rightarrow CONSERVATION OF
ANGULAR MOMENTUM

$U(1)$ \rightarrow GROUP OF 1-D ROTATIONS $e^{i\alpha}$

$SU(2)$ \rightarrow GROUP OF ALL 2×2 MATRICES
(WITH UNIT DETERMINANT)
 $\begin{pmatrix} e & \\ & y \end{pmatrix}$ \rightarrow SYMMETRY OF WEAK FORCE

$SU(3)$ \rightarrow 3×3

SYMMETRY OF
COLOUR FORCE

$U = e^{i\epsilon F}$ ACTION ON WAVE FUNCTION

$$\psi \rightarrow U\psi = e^{i\epsilon F}\psi = \left(1 + i\epsilon F + \frac{(i\epsilon F)^2}{2!} + \dots\right)\psi$$

GENERALLY $e^{i\epsilon F} \neq e^{i\epsilon F^\dagger}$ SINCE $F \neq F^\dagger$

U IS NOT GENERALLY HERMITIAN \rightarrow NOT OBSERVABLE

IF U IS UNITARY $\rightarrow U^\dagger U = 1$

$$e^{-i\epsilon F^\dagger} e^{i\epsilon F} = 1$$

$$e^{i\epsilon(F - F^\dagger)} = 1$$

$F = F^\dagger \rightarrow$ SO F , THE GENERATOR OF U IS HERMITIAN

\nexists OBSERVABLE

U UNITARY \rightarrow GENERATOR F OBSERVABLE

ASSUME U IS ALSO SYMMETRY OPERATOR $\rightarrow [H, U] = 0$

$$U = e^{i\varepsilon F} \rightarrow U = 1 + i\varepsilon F \text{ FOR } \varepsilon F \ll 1$$

$$\begin{aligned} [H, U] &= HU - UH \\ &= H(1 + i\varepsilon F) - (1 + i\varepsilon F)H \\ &= H + i\varepsilon HF - H - i\varepsilon FH \\ &= i\varepsilon HF - i\varepsilon FH \\ &= i\varepsilon [H, F] \end{aligned}$$

$$\text{IF } [H, U] = 0 \rightarrow [H, F] = 0$$

U IS A SYMMETRY
OF THE HAMILTONIAN

\Rightarrow GENERATOR F IS
A CONSERVED
OBSERVABLE

SYMMETRY

SPACE TRANSLATIONS
ROTATIONS
TIME
ROTATION IN ISOSPACE

?
?
?

GLOBAL $SU(3)$ GAUGE
GLOBAL $SU(2)$ GAUGE
GLOBAL $U(1)$ GAUGE
LOCAL $U(1)$ GAUGE
LOCAL $SU(2)$ GAUGE
LOCAL $SU(3)$ GAUGE

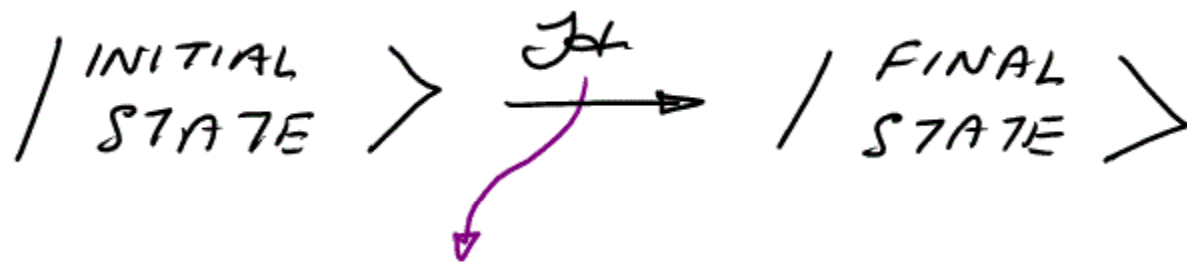
CONSERVED QUANTITY

MOMENTUM
ANGULAR MOMENTUM
ENERGY
ISOSPIN
LEPTON FLAVOR
QUARK FLAVOR
BARYON NUMBER
COLOUR CHARGE
WEAK CHARGE
ELECTRIC CHARGE
EM FORCE
WEAK FORCE
COLOUR FORCE

GENERAL HAMILTONIAN FOR PARTICLES
LAGRANGIAN DENSITY IN FIELD THEORY

$$\mathcal{H} = \mathcal{H}_{\text{COLOR}} + \mathcal{H}_{\text{EM}} + \mathcal{H}_{\text{WEAK}} + \mathcal{H}_{\text{GRAVITY}}$$

EACH $\mathcal{H} \rightarrow$ DIFFERENT SYMMETRIES \rightarrow DIFFERENT CONSERVATION LAWS



DETERMINES WHICH QUANTUM NUMBERS CONSERVED

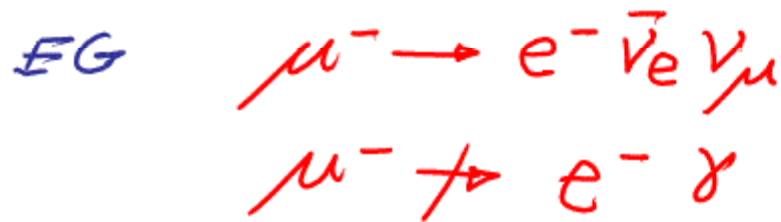
DETERMINES WHICH

\mathcal{H} ACTING

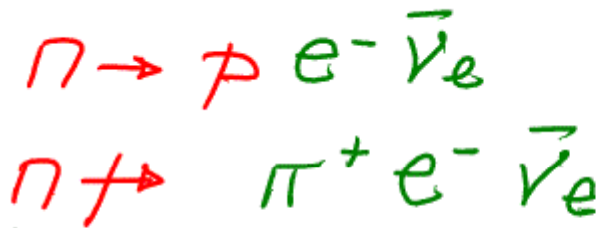
MEASURING
CONSERVED QUANTUM
NUMBER

DETERMINE
SYMMETRIES
OF HAMILTONIAN

DEFINE FORM
OF
HAMILTONIAN



LEPTON NUMBER
CONSERVATION IS A
SYMMETRY OF H_{wk}



DOES CONSERVATION
OF BARYON NUMBER
IMPLY

$$[H_{strong}, B] = 0$$

DEPENDS ON
WHETHER WE

SEE

EXPERIMENT!

BARYON #
OPERATOR

CONSERVATION OF ELECTRIC CHARGE

$$e^- \rightarrow \nu \gamma \quad (\tau > 2 \times 10^{21} \text{ years})$$

CONSERVATION OF ELECTRIC CHARGE \rightarrow SYMMETRY

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$Q \rightarrow$ AN OPERATOR CORRESPONDING TO "ELECTRIC" CHARGE

$\langle Q \rangle$ CONSERVED IF $[\hat{H}, Q] = 0$

ψ CAN BE AN EIGNFUNCTION OF BOTH ENERGY AND CHARGE

$$Q \psi = q \psi$$

PHYSICS IS INVARIANT UNDER PHASE TRANSFORMATION

$$\psi \rightarrow e^{i\varepsilon Q} \psi$$

GLOBAL GAUGE (PHASE) TRANSFORMATION
→ INDEPENDANT OF \vec{x}, t

$$\psi' = e^{i\varepsilon Q} \psi$$
$$\psi \psi^* = (\psi')^* \psi' \quad \text{PROBABILITIES UNCHANGED}$$

GAUGE INVARIANCE MEANS THAT

$$i\hbar \frac{d\psi'}{dt} = \mathcal{H} \psi'$$

SAME
DYNAMICS

JUST AS!

$$i\hbar \frac{d\psi}{dt} = \mathcal{H} \psi$$

$$i\hbar \frac{d\psi'}{dt} = \mathcal{H} \psi' \quad \text{REWRITE IN TERMS OF UNTRANSFORMED WAVE FUNCTIONS}$$

$$i\hbar \frac{d}{dt} e^{i\varepsilon Q} \psi = \mathcal{H} e^{i\varepsilon Q} \psi$$

$$i\hbar \frac{d}{dt} \left(\underbrace{e^{-i\varepsilon Q^\dagger}}_{\#} e^{i\varepsilon Q} \right) \psi = e^{-i\varepsilon Q^\dagger} \mathcal{H} e^{i\varepsilon Q} \psi \quad \# \leftarrow \text{MULT}$$

ASSUME Q OBSERVABLE \rightarrow HERMITIAN

$$i\hbar \frac{d\psi}{dt} = \underbrace{e^{-i\varepsilon Q} \mathcal{H} e^{i\varepsilon Q}}_{\leftarrow Q^\dagger = Q} \psi$$

cf $i\hbar \frac{d\psi}{dt} = \mathcal{H} \psi$

$$i\hbar d\psi/dt = e^{-i\epsilon Q} \mathcal{H} e^{i\epsilon Q} \cdot \psi$$

SAME
DYNAMICS

COMPARE TO: $i\hbar d\psi/dt = \mathcal{H} \psi$

THEN $e^{-i\epsilon Q} \mathcal{H} e^{i\epsilon Q} = \mathcal{H}$

EXPAND EXPONENTIAL - INFINITESIMAL GAUGE TRANSFORM $\rightarrow \epsilon Q \rightarrow 0$

$$(1 - i\epsilon Q) \mathcal{H} (1 + i\epsilon Q) = \mathcal{H} \rightarrow 0$$

$$\mathcal{H} + i\epsilon \mathcal{H} Q - i\epsilon Q \mathcal{H} - \cancel{(i\epsilon)^2 Q \mathcal{H} Q} = \mathcal{H}$$

$$[\mathcal{H} Q - Q \mathcal{H}] = \mathcal{H} - \mathcal{H} = 0$$

$\rightarrow [\mathcal{H}, Q] = 0$

\nearrow
Q CONSERVED

$e^{i\epsilon Q}$ IS A SYMMETRY

\Downarrow
Q IS A CONSERVED
QUANTITY.

• GLOBAL GAUGE SYMMETRY \rightarrow CONSERVED CHARGE

• PHYSICS MUST BE INVARIANT UNDER ARBITRARY PHASE CHANGES AT DIFFERENT POINTS IN SPACE-TIME

CHANGES CANNOT BE TRANSMITTED INSTANTANEOUSLY \rightarrow SPECIAL RELATIVITY

$$\psi \rightarrow \psi e^{i\epsilon(\vec{x}, t) q}$$

LOCAL GAUGE INVARIANCE \rightarrow FORCES

A TASTE OF LOCAL GAUGE SYMMETRY

LOCAL GAUGE
INVARIANCE

$$\psi \rightarrow \psi e^{i\alpha(\vec{x}, t)}$$

TOY
MODEL

ARBITRARY CHANGE AT
EVERY SPACE-TIME POINT

ILLUSTRATION \rightarrow JUST SPATIAL DEPENDENCE

$$H \psi(\vec{x}) = E \psi(\vec{x})$$

HAMILTONIAN

$E_{TOTAL} = KINETIC + POTENTIAL$

$$= \frac{P^2}{2m} + V(\vec{x})$$

SO

$$H = \frac{-\hbar^2}{2m} \nabla^2 + V(\vec{x})$$

KINETIC

POTENTIAL

IF $\psi \rightarrow \psi e^{i\alpha(\vec{x})}$ DO WE STILL HAVE
SAME SCHRÖDINGER? \rightarrow INVARIANCE

SINCE $\bar{\nabla} [e^{i\alpha(\vec{x})} \psi(\vec{x})] \neq e^{i\alpha(\vec{x})} \bar{\nabla} \psi(\vec{x})$

$$\text{IN } \left[\frac{-\hbar^2}{2m} \bar{\nabla}^2 + V(\vec{x}) \right] \psi(\vec{x}) = E(\vec{x}) \psi(\vec{x})$$

THE RIGHT & LEFT HAND SIDES BEHAVE DIFFERENTLY
 \rightarrow SO DO NOT HAVE INVARIANCE UNDER

$$\psi \rightarrow e^{i\alpha(\vec{x})} \psi$$

BUT MUST BE SINCE

$$\psi \psi^* = e^{i\alpha(\vec{x})} \psi (e^{i\alpha(\vec{x})} \psi)^*$$

HOW TO ENSURE LOCAL GAUGE INVARIANCE?

POSTULATE THAT HAMILTONIAN IS INCOMPLETE

$$\left[-\frac{\hbar^2}{2m} \bar{\nabla}^2 + V(\bar{x}) \right] \psi(\bar{x}) = E \psi(\bar{x})$$

UNDER $\psi \rightarrow e^{i\alpha(\bar{x})} \psi$ RHS $E \psi(\bar{x}) \rightarrow \overset{\text{A NUMBER}}{\downarrow} E e^{i\alpha(\bar{x})} \psi(\bar{x})$
 $= e^{i\alpha(\bar{x})} E \psi(\bar{x})$

FOR INVARIANCE NEED LHS!

$$\left[-\frac{\hbar^2}{2m} \bar{\nabla}^2 + V(\bar{x}) \right] e^{i\alpha(\bar{x})} \psi(\bar{x}) = e^{i\alpha(\bar{x})} \left[-\frac{\hbar^2}{2m} \bar{\nabla}^2 + V(\bar{x}) \right] \psi(\bar{x})$$

BUT

$$\bar{\nabla}(\bar{x}) [e^{i\alpha(\bar{x})} \psi(\bar{x})] = e^{i\alpha(\bar{x})} [i \bar{\nabla} \alpha(\bar{x}) \psi(\bar{x}) + \bar{\nabla} \psi(\bar{x})]$$
$$\neq e^{i\alpha(\bar{x})} \psi(\bar{x})$$

→ CANNOT BE TRUE → NO LOCAL GAUGE INVARIANCE

INTRODUCE A NEW VECTOR $A(\bar{x})$

DEFINE COVARIANT DIFFERENTIAL

$$\bar{\nabla} \rightarrow \bar{\nabla} - i\bar{A}(\bar{x}) \text{ AND } \bar{A}(\bar{x}) \rightarrow \bar{A}(\bar{x}) + \bar{\nabla}\alpha(\bar{x})$$

UNDER LOCAL GAUGE TRANSFORMATION $\psi(\bar{x}) \rightarrow e^{i\alpha(\bar{x})}\psi(\bar{x})$

WE NOW HAVE A NEW SCHRÖDINGER

$$\left\{ \frac{-\hbar^2}{2m} \left(\bar{\nabla} - i\bar{A}(\bar{x}) \right)^2 + V(\bar{x}) \right\} \psi(\bar{x}) = E \psi(\bar{x})$$

UNDER LOCAL GAUGE TRANSFORM

$$\left\{ \frac{-\hbar^2}{2m} \left(\bar{\nabla} - i(A(\bar{x}) + \bar{\nabla}\alpha(\bar{x})) \right)^2 + V(\bar{x}) \right\} e^{i\alpha(\bar{x})}\psi(\bar{x}) = e^{i\alpha(\bar{x})} E \psi(\bar{x})$$

LOOK AT THIS PART

$$\begin{aligned}
& (\bar{\nabla} - i \bar{A}(\bar{x}) - i \bar{\nabla} \alpha(\bar{x})) e^{i \alpha(\bar{x})} \psi(\bar{x}) \\
&= \bar{\nabla} \left[e^{i \alpha(\bar{x})} \psi(\bar{x}) \right] - i A(\bar{x}) e^{i \alpha(\bar{x})} \psi - i \bar{\nabla} \alpha(\bar{x}) e^{i \alpha(\bar{x})} \psi(\bar{x}) \\
&= \cancel{i \bar{\nabla} \alpha(\bar{x}) e^{i \alpha(\bar{x})} \psi} + e^{i \alpha(\bar{x})} \bar{\nabla} \psi(\bar{x}) - i A(\bar{x}) e^{i \alpha(\bar{x})} \psi - \cancel{i \bar{\nabla} \alpha(\bar{x}) e^{i \alpha(\bar{x})} \psi(\bar{x})} \\
&= e^{i \alpha(\bar{x})} (\bar{\nabla} - i \bar{A}(\bar{x})) \psi(\bar{x})
\end{aligned}$$

$$(\bar{\nabla} - i A(\bar{x})) \psi(\bar{x}) \xrightarrow{\text{GAUGE}} e^{i \alpha(\bar{x})} (\bar{\nabla} - i A(\bar{x})) \psi(\bar{x})$$

CAN JUST PULL \uparrow THRU \uparrow

$$(\bar{\nabla} - i A(\bar{x}))^2 \psi(\bar{x}) \xrightarrow{\text{GAUGE}} e^{i \alpha(\bar{x})} (\bar{\nabla} - i A(\bar{x}))^2 \psi(\bar{x})$$

INTRODUCING $A(\vec{x})$ AND ITS TRANSFORMATION

$$A(\vec{x}) \rightarrow A(\vec{x}) + \vec{\nabla} \alpha(\vec{x})$$

FORCES GAUGE INVARIANCE

$$\left\{ \frac{-\hbar^2}{2m} \left(\vec{\nabla} - i\vec{A}(\vec{x}) \right)^2 + V(\vec{x}) \right\} \psi(\vec{x}) = E \psi(\vec{x})$$

ORIGINAL
KINETIC
ENERGY

NEW KINETIC ENERGY
DESCRIBES MOTION OF NEW
PARTICLE

↓ "VECTOR POTENTIAL" $A(\vec{x})$

LOCAL GAUGE INVARIANCE \rightarrow E.M. FORCE $\rightarrow \gamma$

REQUIRED SYMMETRY $\psi \rightarrow \underbrace{e^{i\alpha(\vec{x})}}_{\text{GROUP } U(1)} \psi$

$U(1) \rightarrow$ GAUGE SYMMETRY OF EM FORCE

LET'S LOOK AT PUTTING TIME IN

$$\psi(\bar{x}) \rightarrow \psi(\bar{x}, t)$$

$$\alpha(\bar{x}) \rightarrow \alpha(\bar{x}, t)$$

WE CAN ANTICIPATE THAT THE TIME VARIATION OF α WILL HAVE TO BE CANCELLED JUST AS SPACE VARIATION SO WRITE.

$$\left\{ \frac{-\hbar^2}{2m} \left(\bar{\nabla} - i\bar{A}(\bar{x}, t) \right)^2 + v(\bar{x}) \right\} \psi(\bar{x}, t)$$

$$- \hbar \dot{\phi}(\bar{x}, t) \psi(\bar{x}, t) = E \psi(\bar{x})$$

AND DEFINE OUR GAUGE TRANSFORMATION TO MAKE $\dot{\phi}(\bar{x}, t)$ CANCEL TIME VARIATION

$$\left(\vec{\nabla} - iA(\vec{x}, t) \right)^2 + v(x) \left\{ \psi(\vec{x}, t) - \hbar \phi(\vec{x}, t) \psi(\vec{x}, t) \right\} = E \psi$$

$$\text{IF } \psi(\vec{x}, t) \rightarrow e^{i\alpha(\vec{x}, t)} \psi(\vec{x}, t)$$

$$\text{THEN } \phi(\vec{x}, t) \rightarrow \phi(\vec{x}, t) + d\alpha/dt$$

$$A(\vec{x}, t) \rightarrow A(\vec{x}, t) + \vec{\nabla}\alpha(\vec{x}, t)$$

THE $\hbar \phi \psi$ TERM GIVES AN EXTRA $-\hbar \frac{d\alpha}{dt} \psi$ ON LHS

→ CANCELLATION

→ LOCAL GAUGE INVARIANCE

$$A_\mu = (\phi, \vec{A})$$

$$\left\{ -\frac{\hbar^2}{2m} \left(\vec{\nabla} - i\vec{A}(\vec{x}, t) \right)^2 + v(x) \right\} \psi(\vec{x}, t) - \hbar \phi(\vec{x}, t) \psi(\vec{x}, t) = E\psi$$

CLEARLY A_μ IS A 4-VECTOR $A(t, x, y, z)$

x, y, z COMPONENTS OF $A \rightarrow$ SPIN 1
 $(2J+1) = 3$

OUR THEORY HAS $A\psi$ TERMS

NO AA TERMS

IF WE SAY ψ DESCRIBES AN ELECTRON

WE HAVE (STILL A TOY) THEORY

ELECTRON

SPIN 1 PARTICLE INTERACTS WITH ELECTRON

SPIN 1 PARTICLE DOES NOT INTERACT WITH ITSELF

" $A \rightarrow$ EM 4-POTENTIAL"

THERE ARE ALSO GAUGE INVARIANT

$$\frac{dA_x}{dy} - \frac{dA_y}{dx}$$

WHICH CAN BE
REARRANGED INTO A
WAVE EQUATION

$$\nabla^2 A = \frac{1}{c^2} \frac{d^2 A}{dt^2}$$

← WAVE EQUATION FOR
A MASS LESS PARTICLE

GAUGE THEORIES

GAUGE THEORIES HAVE BEEN BUILT FOR

COLOUR FORCE - $SU(3)$

WEAK FORCE - $SU(2)$

EM FORCE - $U(1)$

NON-ABELIAN
(GENERATORS DO NOT
COMMUTE)

ABELIAN

(GENERATORS COMMUTE)

GENERALLY GAUGE INVARIANCE LEADS TO

MASSLESS GAUGE BOSONS

$W^\pm Z^0 ?? \rightarrow$ HIGGS MECHANISM