

LECTURE 18: QCD (Part 1)

Overview:

- QCD introduction
- Feynman rules
- Calculating colour factors
- quark anti-quark annihilation

(I used Quigg and mostly Griffiths as references)

QCD (I)

(2)

Evidence that quarks come in 3 different "colours" include

→ spin-stat. problem for baryons

→ cross section $e^+e^- \rightarrow$ hadrons

→ r branching ratios

→ π^0 lifetime

→ anomaly cancellation

Could we use $U(1)$, $SO(3)$ as candidate gauge groups for the strong interaction?

$SO(3) \rightarrow$ no asymptotic freedom

\rightarrow existence of diquark states, no distinction between quarks and anti-quarks

QCD (II)

$U(3)$ we get a singlet gauge boson \rightarrow long-range interaction.

For the Lagrangian:

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m)\psi - \frac{1}{2} \text{Tr}(G_{\mu\nu} G^{\mu\nu})$$

$$\psi = \begin{pmatrix} \psi \\ \psi \\ \psi \end{pmatrix}$$

$$D_\mu = \partial_\mu + ig B_\mu$$

\rightarrow 3×3 matrix: $B_\mu = \frac{1}{2} \lambda \cdot b_\mu = \frac{1}{2} \lambda^a b_\mu^a$

b_μ : 8 colour gauge fields

$$G_{\mu\nu} = \frac{1}{2} G_{\mu\nu} \cdot \lambda = \frac{1}{2} G_{\mu\nu}^a \lambda^a$$

$$= (ig)^{-1} [D_\nu, D_\mu] = \partial_\nu B_\mu - \partial_\mu B_\nu + ig [B_\nu, B_\mu]$$

QCD (III)

Properties of the λ matrices:

$$\begin{aligned} \text{Tr}(\lambda^a) &= 0 & F_{123} &= 1 & F_{458} &= F_{678} = \sqrt{3}/2 \\ \text{Tr}(\lambda^k \lambda^l) &= 2\delta^{kl} & F_{147} &= F_{246} = F_{345} = F_{516} = F_{637} = 1/2 \\ [\lambda^i, \lambda^k] &= 2i f^{ijk} \lambda^l \end{aligned}$$

$$G_{\mu\nu}^L = \partial_\nu b_\mu^L - \partial_\mu b_\nu^L + g f^{ijk} b_\mu^j b_\nu^k$$

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

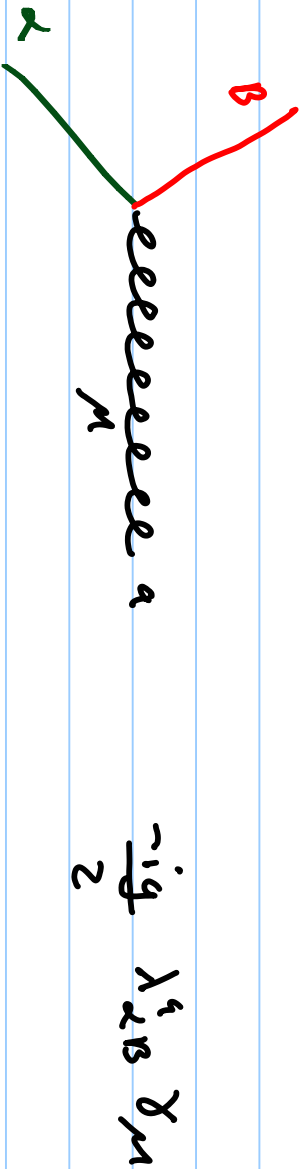
$$\begin{aligned} \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

QCD IV

(5)

quark - gluon interaction: $\mathcal{L}_{int} = -\frac{g}{2} \bar{\psi} \gamma^\mu \psi \lambda^a \gamma_\mu$



→ one gluon exchange force prop. $\frac{g^2}{4} \sum_a \lambda_{ab}^a \lambda_{cd}^a$

$$\alpha + \gamma \rightarrow \beta + \delta$$

Gluons are massless particles with spin 1 which we will represent by pol. vector ϵ^μ .

→ $\epsilon^\mu p_\mu = 0$ (Lorentz Condition)

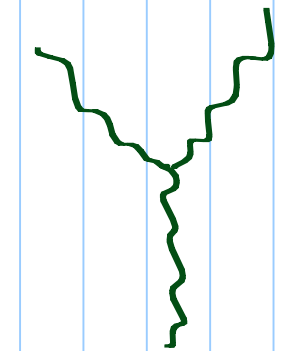
QCD (\mathbb{T})

We will also use Coulomb gauge: $\epsilon^0 = 0$ $\vec{\epsilon} \cdot \vec{p} = 0$

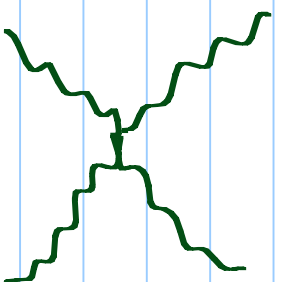
To describe colour state of gluon, we'll use colour vectors:

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv |1\rangle \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv |4\rangle$$

Non-Abelian group \rightarrow gauge mediators interact with themselves:



3-gluon vertex



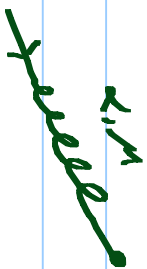
4-gluon vertex

QCD (VII)

⑦

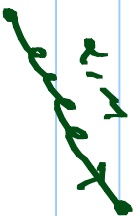
Feynman Rules

incoming gluon:



$$\epsilon_{\mu}(p) a^{\alpha}$$

outgoing gluon:



$$\epsilon_{\mu}^{*}(p) a^{\alpha*}$$

Propagators

quark:



$$\frac{i(\not{q} + m)}{q^2 - m^2}$$

gluon:

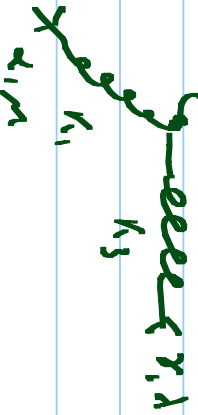


$$-\frac{ig_{\alpha\beta} \delta_{ab}}{q^2}$$

Vertices



$$-\frac{ig_s}{2} \gamma^{\alpha} \gamma_{\mu}$$

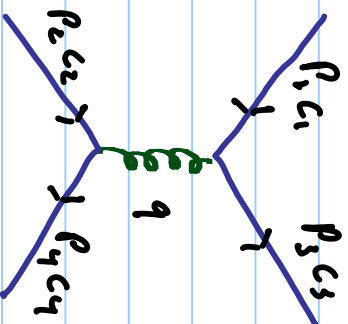


$$-g_s f^{abc} \gamma^{\nu} [g_{\mu\nu} (k_1 - k_2)_{\lambda} + g_{\nu\lambda} (k_2 - k_3)_{\mu} + g_{\lambda\mu} (k_3 - k_1)_{\nu}]$$

QCD

(8)

Consider $u + \bar{d} \rightarrow u + \bar{d}$



$$-iM = [\bar{u}(3)c_3^T] \left[-i \frac{g_s}{2} \lambda^a \gamma^a \right] [u(1)c_1] \left[-i g_{\mu\nu} \frac{\delta^{ab}}{q^2} \right] \\ \times [\bar{d}(2)c_2^T] \left[-i \frac{g_s}{2} \lambda^b \gamma^b \right] [d(4)c_4]$$

$$M = -\frac{g_s^2}{4} \frac{1}{q^2} [\bar{u}(3)\gamma^a u(1)] [\bar{d}(2)\gamma_a d(4)] (c_3^T \lambda^a c_1) (c_2^T \lambda^a c_4)$$

colour Factor

→ same as electron-positron scattering with $g_e \rightarrow g_s$
 but with extra colour factor

Potential similar to EM case $V \propto -\frac{r^i r^j}{r}$

QCD

(7)

— we'll consider the colour octet case e.g. $\bar{r}\bar{b}$

$$c_1 = c_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = c_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$F = \frac{1}{4} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = \frac{1}{4} \lambda_{11}^2 \lambda_{22}^2$$

→ λ^3 and λ^6 are the only relevant matrices

$$F = \frac{1}{4} (\lambda_{11}^3 \lambda_{22}^3 + \lambda_{11}^6 \lambda_{11}^6) = -1/6$$

— now we consider the singlet state: $\frac{1}{\sqrt{3}} (r\bar{r} + b\bar{b} + g\bar{g})$

$$F = \frac{1}{4} \cdot \frac{1}{\sqrt{3}} \left[\left(c_3^T \lambda^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda^2 c_4 \right] + \left(c_3^T \lambda^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda^2 c_4 \right] + \left(c_3^T \lambda^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda^2 c_4 \right] \right]$$

QCD

(11)

quark - quark interaction

$$M = -\frac{g_s^2}{4} \frac{1}{f^2} [\bar{u}(1)\gamma^\mu u(1)] [\bar{u}(2)\gamma^\mu u(2)] (c_1^\dagger \lambda^a c_1) (c_2^\dagger \lambda^a c_2)$$

$$F = \frac{1}{4} (c_1^\dagger \lambda^a c_1) (c_2^\dagger \lambda^a c_2)$$

we get a triplet:

$$\left. \begin{array}{l} (rb - br) / \sqrt{2} \\ (bs - sb) / \sqrt{2} \\ (gr - rg) / \sqrt{2} \end{array} \right\} \text{anti-sym combinations}$$

and a sextet:

$$r, g, b, b$$

+ symmetric combinations

QCD

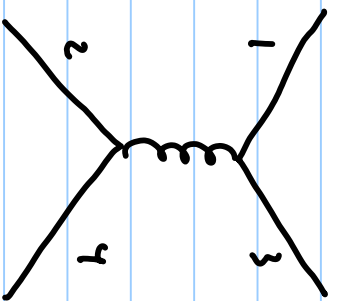
(12)

setlet : $C_1 = C_2 = C_3 = C_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (for r)

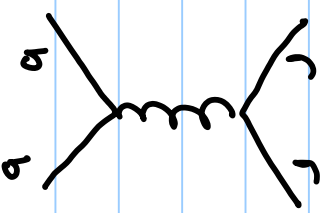
$$F = \frac{1}{4} \left[C_{1001} \lambda^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \left[C_{1100} \lambda^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \frac{1}{4} \lambda_{11}^2 \lambda_{11}^2$$

$$= \frac{1}{4} \left[\lambda_{11}^3 \lambda_{11}^3 + \lambda_{11}^8 \lambda_{11}^8 \right] = \frac{1}{4} \left[1 + \frac{1}{3} \right] = \frac{1}{3}$$

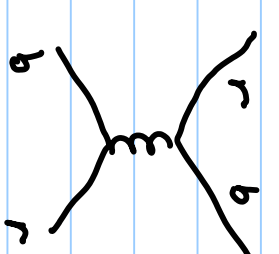
For triplet: For example $(rb-br)/\sqrt{2}$



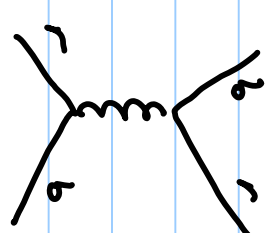
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QCD

(13)

We get:

$$\begin{aligned} F &= \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \left[\left[(100) \lambda^{\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[(010) \lambda^{\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \right. \\ &\quad - \left[(010) \lambda^{\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[(100) \lambda^{\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \\ &\quad + \left[(010) \lambda^{\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \left[(100) \lambda^{\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &\quad \left. - \left[(100) \lambda^{\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \left[(010) \lambda^{\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \right] \\ &= \frac{1}{6} \left[\lambda_{11}^{\alpha} \lambda_{22}^{\alpha} - \lambda_{21}^{\alpha} \lambda_{12}^{\alpha} + \lambda_{22}^{\alpha} \lambda_{11}^{\alpha} - \lambda_{12}^{\alpha} \lambda_{21}^{\alpha} \right] \\ &= \frac{1}{4} \left[\lambda_{11}^{\alpha} \lambda_{22}^{\alpha} - \lambda_{12}^{\alpha} \lambda_{21}^{\alpha} \right] \\ &= \frac{1}{4} \left[\lambda_{11}^3 \lambda_{22}^3 + \lambda_{11}^6 \lambda_{22}^6 - \lambda_{11}^1 \lambda_{21}^1 - \lambda_{12}^2 \lambda_{21}^2 \right] = -\frac{2}{3} \end{aligned}$$

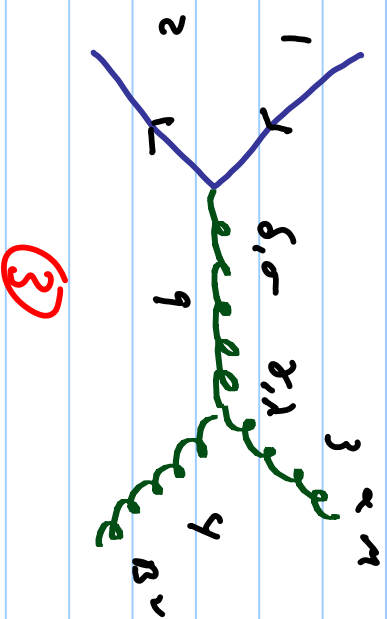
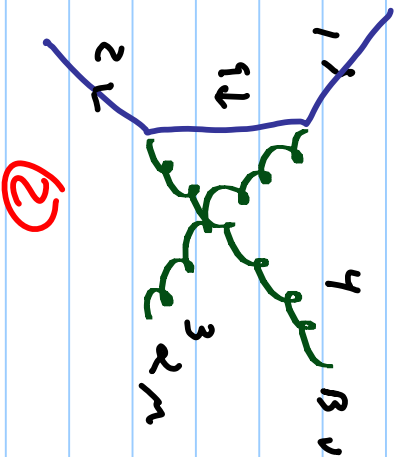
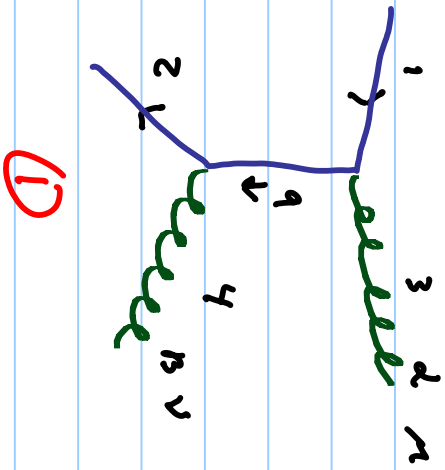
QCD

$$V_{qq}(r) = -\frac{2}{3} \frac{d_s}{r} \quad \text{triplet}$$

$$V_{qq}(r) = \frac{1}{3} \frac{d_s}{r} \quad \text{sextet}$$

not seen ...

Pair ANNIHILATION



(14)

$$\textcircled{1} : -iM_1 = \bar{v}(2) c_2^+ \left[-i \frac{g_s}{2} \lambda^a \gamma^\nu \right] \left[\epsilon_{\nu\mu}^a c_4^* \right] \left[\frac{i(g + m)}{q^2 - m^2} \right] \times \left[-i \frac{g_s}{2} \lambda^a \gamma^\mu \right] \left[\epsilon_3^a c_4^* \right] v(1) c_1$$

QCD

(15)

$$q = p_1 - p_3 \rightarrow$$

$$q^2 - m^2 = p_1^2 - 2 p_1 \cdot p_3 + p_3^2 - m^2 = -2 p_1 \cdot p_3$$

$$M_1 = -\frac{g_s^2}{8} \frac{1}{p_1 \cdot p_3} \bar{v}(2) \left[\cancel{\not{q}} (\cancel{p_1} - \cancel{p_3} + \cancel{m}) \cancel{\not{q}} \right] v(1) \\ \times a_3^\lambda a_4^\rho (c_2^T \lambda^\rho \lambda^\lambda c_1)$$

Diagram 2: switch $3 \leftrightarrow 4$

colour factor: $a_3^\lambda a_4^\rho (c_2^T \lambda^\lambda \lambda^\rho c_1)$

$$(3): -i M_3 = \bar{v}(2) c_2^T \left[-i \frac{g_s}{2} \lambda^\rho \gamma_\rho \right] v(1) c_1 \left[-i g \frac{\cancel{\not{q}} \cancel{\not{q}}}{q^2} \right]$$

$$\cdot \left[-g_s \cancel{F}^{\lambda\beta\delta} \left[g_{\mu\nu} (-\cancel{p}_3 + \cancel{p}_4)_\lambda + g_{\nu\lambda} (-\cancel{p}_4 - \cancel{q})_\mu + g_{\lambda\mu} (\cancel{q} + \cancel{p}_3)_\nu \right] \right] \\ \cdot \epsilon_3^\mu \epsilon_3^\lambda \epsilon_4^\nu a_4^\beta \quad (A)$$

QCD

(16)

$$q^1 = p_3 + p_4, \quad q^2 = 2p_3 \cdot p_4, \quad \epsilon_3 \cdot p_3 = 0 = \epsilon_4 \cdot p_4$$

$$M_3 = i \frac{g_s^2}{4} \frac{1}{p_3 \cdot p_4} \bar{v}(z) [(\epsilon_3 \cdot \epsilon_4) (p_4 - p_3) + 2(p_3 \cdot \epsilon_4) \epsilon_3 - 2(p_4 \cdot \epsilon_3) \epsilon_4] u(z) \\ \times \text{Feynman rules } a_3^\dagger a_4^b (c_2^\dagger \lambda^c c_1) \quad (13)$$

Problem SET #3, problem 2: show how we get (13) from (14) (Griffiths)

To simplify things, we'll assume that the particles are at rest.

$$p_1 = p_2 = (m, 0), \quad p_3 = (E, p), \quad p_4 = (E, -p)$$

$$p_1 \cdot p_3 = p_1 \cdot p_4 = m^2, \quad p_3 \cdot p_4 = 2m^2$$

$$p_3 \cdot \epsilon_4 = -p \cdot \epsilon_4 = -p_4 \cdot \epsilon_4 = 0 \quad (\text{same for } p_4 \cdot \epsilon_3)$$

QCD

(17)

We now get a Total amplitude :

$$M = \frac{-g_s^2}{8m^2} a_3^2 a_4^2 \bar{v}(2) c_2^T \left[\cancel{\not{\epsilon}_3} \not{\epsilon}_4 \not{p}_4 \cancel{\not{\lambda}^2} \not{\lambda}^0 + \cancel{\not{\epsilon}_4} \not{\epsilon}_3 \not{p}_3 \cancel{\not{\lambda}^2} \not{\lambda}^0 \right. \\ \left. - i (\cancel{\epsilon}_3 \cdot \epsilon_4) (\cancel{p}_4 - p_3) \cancel{F_{203}} \not{\lambda}^Y \right] c_1 v(1)$$

orient coordinates with \vec{p} along z :

$$p_3 = m(\gamma^0 - \gamma^3), \quad p_4 = m(\gamma^0 + \gamma^3), \quad (p_4 - p_3) = 2m\gamma^3$$

using

$$\epsilon_3 \epsilon_4 = -(\epsilon_3 \cdot \epsilon_4) - i(\epsilon_3 \times \epsilon_4) \cdot \underline{\Sigma} \quad ; \quad \underline{\Sigma} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}$$
$$\cancel{\not{\epsilon}_4} \not{\epsilon}_3 = -(\cancel{\epsilon}_3 \cdot \epsilon_4) + i(\cancel{\epsilon}_3 \times \epsilon_4) \cdot \underline{\Sigma} \quad ; \quad !$$

we get

$$M = \frac{g_s^2}{8m} a_3^2 a_4^2 \bar{v}(2) c_2^T \left[(\epsilon_3 \cdot \epsilon_4) \{ \not{\lambda}^2, \not{\lambda}^0 \} \gamma^0 \right. \\ \left. + i(\cancel{\epsilon}_3 \times \epsilon_4) \cdot \underline{\Sigma} \left[\not{\lambda}^2, \not{\lambda}^0 \right] \gamma^0 + \{ \not{\lambda}^2, \not{\lambda}^0 \} \gamma^3 \right] c_1 v(1)$$

\rightarrow same as QED result with $\lambda=1$, no colour states

QCD

Now we put the quarks in singlet state (spin 0)

$$M = (M_{\uparrow\downarrow} - M_{\downarrow\uparrow}) / \sqrt{2}$$

For $M_{\uparrow\downarrow}$ we have: $\bar{v}(2) \gamma^0 v(1) = \bar{v}(2) \sum \gamma^0 v(1) = 0$

$$\bar{v}(2) \sum \gamma^3 v(1) = -2m \hat{z}$$

$$M_{\downarrow\uparrow} = -M_{\uparrow\downarrow}$$

$$M = -i\sqrt{2} \frac{g_s^2}{4} (\epsilon_3 \times \epsilon_4)_z g_3^2 g_4^B (c_2^T \{ \lambda^4, \lambda^0 \} c_1)$$

$$\text{colour factor} = \frac{1}{9} g_3^2 g_4^B (c_2^T \{ \lambda^4, \lambda^0 \} c_1)$$

Problem set 3, problem #3

if quarks occupy colour singlet state $\frac{1}{\sqrt{3}} (r\bar{r} + b\bar{b} + g\bar{g})$

Find the colour factor.

QCD

(19)

$$M = -4 \cdot \sqrt{2/3} g_s^2$$

$$\begin{aligned} \rightarrow \frac{d\sigma}{d\Omega} &= \left[\frac{1}{8\pi(E_1 + E_2)} \right]^2 \frac{|p_{e1}|}{|p_{i1}|} |M|^2 \\ &\Rightarrow \sigma \propto \frac{2}{5} g_s^2 \quad * \end{aligned}$$

PS 4, problem 1: 9.14 of Griffiths