LECTURE 2: Calculation of QED Cross Sections and Decay Rates (Review Part I)

Overview:

- Experimental considerations
- Transition rates
- Spinless electron-muon scattering
- Cross section and decay rate calculations

(Mostly follows Halzen and Martin. I recommend Burgess and Moore for a more thorough and rigourous treatment)

What do we measure and what do we need to calculate?

-We want to determine various properties of particles and their interactions.

-A common technique is to collide particles at a very high rate and at very high energies. We then look at the products of these collisions. A few notes:

-What we end up observing are the long-lived decay products of what was initially produced in the collision

-There can be a lot of decay products and we need to determine how to relate these to what we initially produced

-The probablility of producing what we are interested in

What do we measure? (see ppt slides)

What do we want to calculate?

-we want to determine a quantity related to the probablility of producing certain final states that is independent of the rate of the collisions. This quantity should not involve time.

Let's start from here:

Particle in a box of volume V

Free-particle equation: $H\varphi_n = E_n \varphi_n$

$$\int_V \varphi_m^* \varphi_n \, d^3x = \delta_{mn}$$

equation with interaction potential:

$$(H + V(x,t))\psi = i \frac{\partial \psi}{\partial t}$$

Solutions to $(H + V(x,t))\psi = i\frac{\partial \psi}{\partial t}$ can be expressed as

$$\psi = \sum_n a_n(t)\, \varphi_n(x)\, e^{-iE_n t} \qquad (1)$$

- want to find the $a_n(t)$

insert (1) into Schrödinger equation:

$$\sum_n E_n a_n \cancel{\varphi_n(x)}\, e^{-iE_n t} + \sum_n V(x,t)\, a_n \varphi_n(x)\, e^{-iE_n t}$$

$$= i\sum \frac{da_n}{dt} \varphi_n(x)\, e^{-iE_n t} + i\sum a_n \cancel{\varphi_n(x)} \cdot -iE_n e^{iE_n t}$$

- multiply both sides by φ_f^*, integrate over V

(2) LHS: $i\sum_n \int \frac{da_n}{dt} \varphi_f^* \varphi_n\, e^{-iE_n t}\, d^3x = i\cancel{\sum_n} \frac{da_f}{dt} e^{-iE_f t}$

(3) RHS: $\sum_n a_n(t) \int \varphi_f^* V \varphi_n\, d^3x\, e^{-iE_n t}$

From 2 and 3, we get

$$\frac{da_f}{dT} = -i \sum_n a_n(t) \int \varphi_f^* V \varphi_n \, e^{i(E_f - E_n)T} d^3x$$

→ before potential act on particle, it is in eigenstate i at $T = -t/2$

$$a_i(-T/2) = 1 \quad , \quad a_n(-T/2) = 0 \qquad i \neq n$$

$$\rightarrow \quad \frac{da_f}{dT} = -i \int d^3x \, \varphi_f^* V \varphi_i \, e^{i(E_f - E_i)T}$$

→ assume potential is small and transient and integrate

$$a_f(t) = -i \int_{-t/2}^{t} dT' \int d^3x \, \varphi_f^* V \varphi_i \, e^{i(E_f - E_i)T'}$$

At Time $+T/2$ when interactions have ceased:

$$a_f(T/2) \equiv T_{fi} =$$

$$-i \int_{-T/2}^{T/2} dT' \int d^3x \left[\varphi_f(x) e^{-iE_f t} \right]^* V(x,t) \left[\varphi_i(x) e^{-iE_i T} \right]$$

in covariant form:

$$T_{fi} = -i \int d^4x \; \varphi_f^*(x) V(x) \varphi_i(x)$$

good approx. if $a_f(t)$ is small.

Is $|T_{fi}|^2$ the probability that a particle in state i ends up in state f under the influence of V?

Let's take $V(x,t) = V(x)$, we get

$$T_{fi} = \underbrace{-i\int d^3x\, \varphi_f(x)\, V(x)\, \varphi_i(x)}_{V_{fi}} \int_{-\infty}^{\infty} dt\; e^{i(E_f - E_i)t}$$

$$= -2\pi i\, \underbrace{\delta(E_f - E_i)}\, V_{fi}$$

↳ energy conservation during transition

→ uncertainty principle implies that Δt between i and f = ∞

→ $|T_{fi}|^2$ not useful ...

We try transition prob. per unit time:

$$W = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$$

We need to square T_{fi}.

remember that $T_{fi} = -i V_{fi} \int_{-\infty}^{\infty} dT \, e^{i(E_f - E_i)T}$

$$= -2\pi i V_{fi} \, \delta(E_f - E_i)$$

$$W = \lim_{T \to \infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{T/2} dT \, e^{i(E_f - E_i)T}$$

applied lim already

$$= \lim_{T \to \infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{T/2} dT$$

$$= 2\pi |V_{fi}|^2 \delta(E_f - E_i)$$

→ need to deal with initial and final states. Here we'll start with well-defined initial state ending up in a set of final states let $\rho(E_f)$ be density of final states

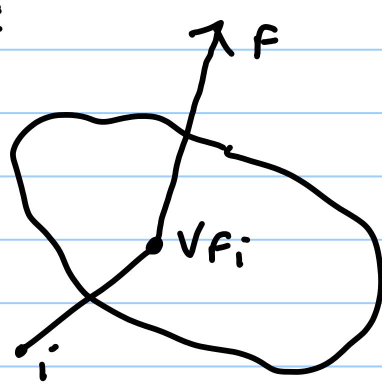
$\rightarrow$ $\rho(E_f)\,dE_f$ is # of final states in energy interval $[E_f, E_f + dE_f]$

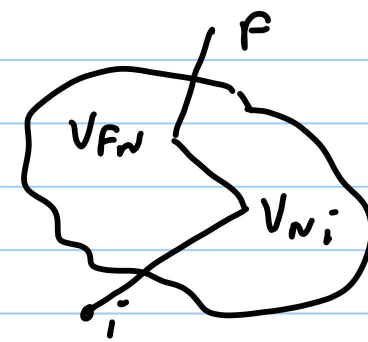
so: $$W_{fi} = 2\pi \int dE_f\, \rho(E_f)\, |V_{fi}|^2\, \delta(E_f - E_i)$$

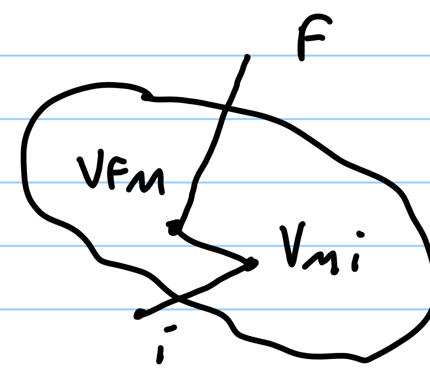
$$= 2\pi\, |V_{fi}|\, \rho(E_i)$$ $\rightarrow$ Transition prob. per unit Time

$\rightarrow$ Fermi's Golden Rule

Note that the result we obtained was an approx. which we can try to improve.

First order: 

second: 

+ 

+ ...

We go back and insert:

$$a_f(t) = -i\int_{-t/2}^{t} dt' \int d^3x\, \varphi_f^* V \varphi_i\, e^{i(E_f - E_i)t'}$$

which becomes: $-i\int_{-t/2}^{t} dt' \int d^3x\, \varphi_n^* V \varphi_i\, e^{i(E_n - E_i)t'}$

$f \to n$

here

$$\frac{da_f}{dT} = -i \sum_n a_n(t) \int \varphi_f^* V \varphi_n\, e^{i(E_f - E_n)T}\, d^3x$$

and obtain:

$$\frac{da_f}{dt} = \underbrace{-i\int d^3x\, \varphi_f^* V \varphi_i\, e^{i(E_f - E_i)T}}_{\text{First order result above}} \quad \text{or} \quad -iV_{fi}\, e^{i(E_f - E_i)t}$$

$$+ (-i)^2 \left[\sum_{n \neq i} V_{ni} \int_{-T/2}^{T} dT'\, e^{i(E_n - E_i)T'} \right] V_{fn}\, e^{i(E_f - E_n)T}$$

integrate to get T_{fi} ($\equiv a_f(t)$ when interactions have ceased)

(4) $T_{fi} = \underbrace{\ldots}_{\text{first order}} + -\sum_{n \neq i} V_{fn} V_{ni} \int_{-\infty}^{\infty} dt\, e^{i(E_f - E_n)t} \int_{-\infty}^{t} dt'\, e^{i(E_n - E_i)t'}$

use $\int_{-\infty}^{t} dt'\, e^{i(E_n - E_i - i\varepsilon)t'} = \dfrac{i\, e^{i(E_n - E_i - i\varepsilon)t}}{E_i - E_n + i\varepsilon}$

insert into (4)

$$T_{fi} = \ldots -2\pi i \sum_{n \neq i} \frac{V_{fn} V_{ni}}{\underbrace{E_i - E_n + i\varepsilon}_{\text{propagator term } \frac{1}{(E_i - E_n)}}} \delta(E_f - E_i)$$

note that V_{ni} is $\int d^3x\, \varphi_n V \varphi_i$ "vertex factor"

For V, we'll next consider EM interactions

← open parenthesis

Gauge Invariance and E and M.

Why "gauge"?

Weyl was looking for geometric basis for both E and M and gravity by considering a space-time dependent change of scale.

Eich invarianz: gauge or standard of calibration

consider $F(x)$ that changes between x_μ and $(x_\mu + dx_\mu)$

if space is uniform

$$F(x+dx) = F(x) + \partial^\mu F \, dx_\mu$$

What if the unit of measure or calibration changes in going from x_μ to $(x_\mu + dx_\mu)$?

$$F(x+dx) = (F(x) + \partial^\mu F \, dx_\mu) \cdot (1 + S^\nu dx_\nu)$$

$$F(x+dx) = (F(x) + \partial^\mu F\, dx_\mu)\cdot(1 + S^\nu dx_\nu)$$

$$= F(x) + (\partial^\mu F(x) + F(x) S^\mu)\, dx_\mu + O(dx)^2$$

$$\Delta F = \underbrace{(\partial^\mu + S^\mu)}_{\text{modified differential operator}} F\, dx_\mu \qquad \text{(First order)}$$

For E and M

$$p_\mu = (E, p_x, p_y, p_z)$$

From QM $\quad p^\mu \to i\partial^\mu \qquad (i\partial^0, -i\nabla)$

$$(p^\mu - eA^\mu) \to i(\partial^\mu + ieA^\mu)$$

if $\quad S^\mu \to ieA^\mu$

$$(1 + ieA^\mu dx_\mu) \simeq e^{ieA^\mu dx_\mu}$$

$\rightarrow$ invariance under change of phase (Kept gauge invariance)

Phase invariance in QM

$$\langle O \rangle = \int \psi^* O_{op} \psi$$

$\rightarrow$ invariant under overall phase $\psi(x) \rightarrow e^{i\alpha}\psi(x)$

but, relative phases do matter

Can we formulate QM with position dependent (local) phase rotations?

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$$

Yes, but at a price

$$\partial_\mu \psi \rightarrow \partial_\mu \psi' = e^{i\alpha(x)}\left[\partial_\mu \psi(x) + i(\partial_\mu \alpha(x))\psi(x)\right]$$

picked up extra term ...

replace $\partial_\mu \rightarrow D_\mu \equiv \partial_\mu - ieA_\mu$

with $A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$

then

$$D_\mu \psi(x) \rightarrow e^{i\alpha(x)} D_\mu \psi(x)$$

quantities like $\psi^* D_\mu \psi(x)$ are invariant under local phase transformations

what did we do? introduced a current

had free field theory $\rightarrow$ turned on EM interactions

$i\partial^\mu \rightarrow i\partial^\mu + eA^\mu$ at the centre of QED

note that a mass term $m A^\mu A_\mu$ would not preserve local gauge invariance

Klein Gordon Equation

we can start with: $E^2 - p^2c^2 = m^2c^4$

$$p^\mu p_\mu - m^2c^2 = 0$$

$p^\mu \to i\hbar\partial^\mu$: $-\hbar^2 \partial^\mu \partial_\mu \varphi - m^2c^2\varphi = 0$

$\hbar = c = 1$ from now on...

$$(\partial_\mu \partial^\mu + m^2)\varphi = 0$$

Turn on QED : $i\partial^\mu \to i\partial^\mu + eA^\mu$

$\partial^\mu \to \partial^\mu - ieA^\mu$

$= \quad (\partial_\mu \partial^\mu + m^2)\varphi = -V\varphi$

$V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2A^2$

$\alpha = \frac{e^2}{4\pi} \simeq 1/137 \quad \to$ small $\to$ pert. theory

lowest order, omit e^2 Term] close parenthesis

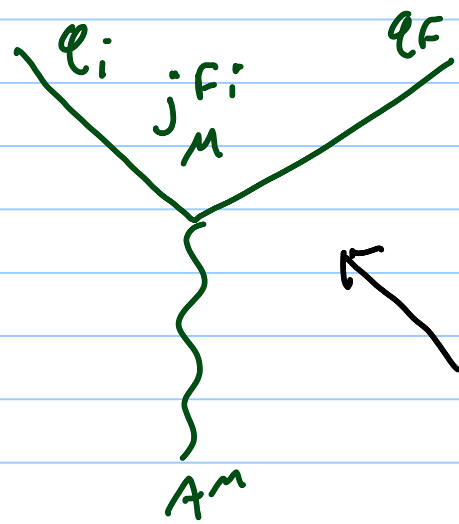
Back To Transition Amplitudes

$$T_{fi} = -i \int \varphi_f^*(x)\, V(x)\, \varphi_i(x)\, d^4x$$

$$= i \int \varphi_f^* \; ie \left(A^\mu \partial_\mu + \underbrace{\partial_\mu}_{\text{acts on } A^\mu \text{ and } \varphi_i} A^\mu \right) \varphi_i \, d^4x$$

int. by parts: $\int \varphi_f^* \partial_\mu (A^\mu \varphi_i) = -\int \partial_\mu (\varphi_f^*) A^\mu \varphi_i$

note $uv\Big|_{-\infty}^{\infty} = 0$ because potential vanishes at ∞



$$T_{fi} = -i \int j_\mu^{fi} A^\mu \, d^4x$$

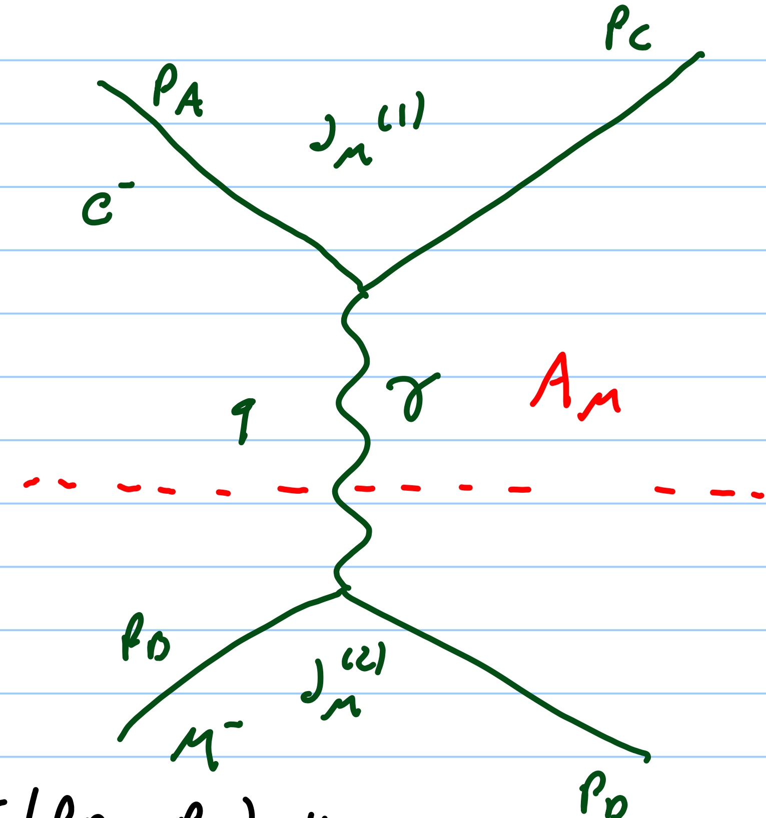
$$j_\mu^{fi}(x) = -ie \left(\varphi_f^* (\partial_\mu \varphi) - (\partial_\mu \varphi_f^*) \varphi_i \right)$$

spinless electron

$$\varphi_i(x) = N_i\, e^{-ip_i\cdot x} \quad , \quad \varphi_f(x) = N_f\, e^{-ip_f\cdot x}$$

$$j_\mu^{fi} = -e N_i N_f (p_i + p_f)_\mu\, e^{i(p_f - p_i)\cdot x}$$

- Electron - Muon Scattering



→ associate A_μ with its source

→ Solve Maxwell equation in Lorentz gauge $(\partial^\mu A_\mu = 0)$

$$\underbrace{\Box^2}_{\partial^\mu \partial_\mu} A^\mu = j^\mu_{(2)}$$

Now: $j^\mu_{(2)} = -e N_B N_D (p_D + p_B)^\mu\, e^{i(p_D - p_B)\cdot x}$

Now: $j^{\mu}_{(2)} = -e N_B N_D (p_D + p_B)^{\mu} e^{i(p_D - p_B)\cdot x}$

Because $\quad \Box^2 e^{iq\cdot x} = -q^2 e^{iq\cdot x}$

with $\quad q = (p_D - p_B)$

$$\rightarrow \quad A^{\mu} = -\frac{1}{q^2} j^{\mu}_{(2)}$$

First order this gives:

$$T_{fi} = -i \int j^{(1)}_{\mu}(x) \left(\frac{-1}{q^2}\right) j^{\mu}_{(2)}(x)\, d^4x$$

$$T_{fi} = -i N_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_B - p_A) \cdot \mathcal{M}$$

with $\quad -i\mathcal{M} = \left(ie(p_A + p_C)^{\mu}\right) \left(\frac{-i g_{\mu\nu}}{q^2}\right) \left(ie(p_B + p_D)^{\nu}\right)$

What do we do with that?

Cross Section Calculation

Step 1 : Fix the normalization

$$\varphi = N e^{-ip\cdot x}$$

prob. density $\rho = 2E|N|^2$

$$\boxed{\rho = i\left(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}\right), \text{ obtained from KG equation}}$$

normalize to $2E$ particles in volume V:

$$\int_V \rho \, dV = 2E \quad \longrightarrow \quad N = \frac{1}{\sqrt{V}}$$

Step 2 : obtain transition rate per unit volume

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

with $T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)\mathcal{M}$

As before, after squaring we get:

$$W_{fi} = (2\pi)^4 \frac{\delta^{(4)}(p_C + p_D - p_A - p_B)\,|\mathcal{M}|^2}{V^4}$$

Step 3: obtain cross section from W_{fi}:

$$\text{cross section} = \frac{W_{fi}}{\text{initial flux}} \cdot \text{number of final states}$$

next page

$$= \frac{V\,d^3p}{(2\pi)^3}$$

$$\boxed{\text{From particle in 1D box we have } p_x L = 2\pi N \text{ or } N = \frac{p_x L}{2\pi}}$$

$$\frac{\#\text{ of final states}}{\text{particle}} = \frac{V\,d^3p}{(2\pi)^3 \cdot 2E}$$

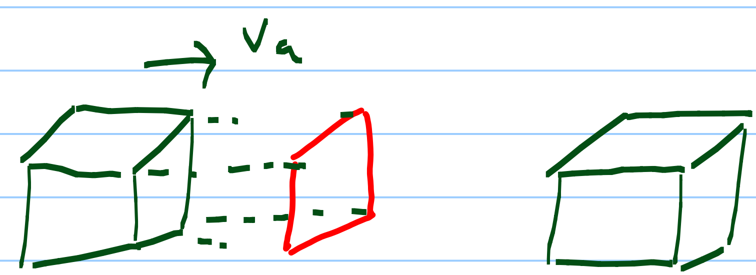
$\Rightarrow$ # of final states : $\dfrac{V\, d^3 p_c}{(2\pi)^3\, 2E_c} \dfrac{V\, d^3 p_D}{(2\pi)^3\, 2E_D}$

initial flux:

# of beam particles passing through unit area per unit time :

$$|v_A|\, 2E_A / V$$

# of target particles per unit volume is : $\dfrac{2E_B}{V}$



Initial flux : $|v_A| \dfrac{2E_A}{V} \dfrac{2E_B}{V}$

Putting everything Together:

$$d\sigma = \frac{V^2}{|v_A|\, 2E_A\, 2E_B} \frac{1}{V^4} |M|^2 \frac{(2\pi)^4}{(2\pi)^6} \delta^{(4)}(p_C + p_D - p_A - p_B) \frac{d^3p_C}{2E_C} \frac{d^3p_D}{2E_D} V^2$$

$\rightarrow V$ goes away [we normalize To 2E particles / unit volume; use unit Volume from now on]

we can write $d\sigma = \frac{|M|^2}{F} \, dQ$