LECTURE 3: Calculation of QED Cross Sections and Decay Rates (Review Part II)

Overview:

- -Continue formulas for cross section and decay calculations
- -Cross section calculation for spinless muon-electron
- -Including spin (Dirac equation refresher)

(This lecture mostly follows Halzen and Martin Chap. 4-6 and Griffiths Chap. 6-7)

Differential cross section (applications) ②

Last lecture we obtained:

$$d\sigma = \frac{|\mathcal{M}|^2}{F}\, dQ$$

invariant amplitude

Lorentz invariant phase space factor ("dLips")

Incident flux

- we obtained an expression for $\mathcal{M}$: $(e\mu)$

$$-i\mathcal{M} = \left(ie(p_A+p_C)^\mu\right)\left(\frac{-i g_{\mu\nu}}{q^2}\right)\left(ie(p_B+p_D)^\nu\right)$$

- we will now calculate dQ and F for the case where we choose the centre of mass frame i.e.

Find dQ, F for $AB \to CD$ in CM

CM Frame, $e\mu$ scattering

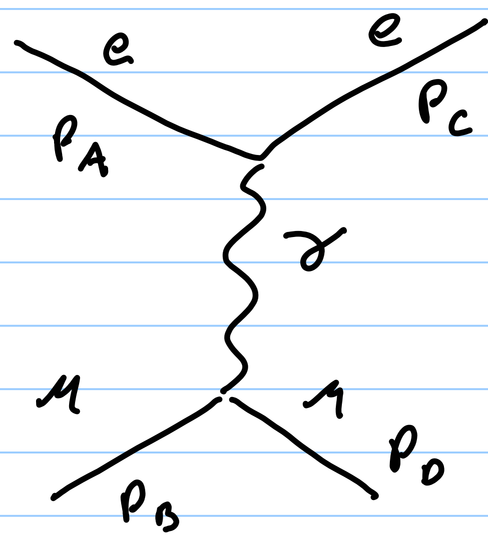
$$dQ = (2\pi)^4\, \delta^4(P_A + P_B - P_C - P_D)\, \frac{d\vec{P}_C}{(2\pi)^3\, 2E_C}\, \frac{d\vec{P}_D}{(2\pi)^3\, 2E_D}$$

$$= \frac{1}{4\pi^2} \frac{d^3\vec{P}_C}{2E_C} \frac{d^3\vec{P}_D}{2E_D}\, \delta^4(P_A + P_B - P_C - P_D)$$

$\hookrightarrow \vec{P}_A = -\vec{P}_B$ in CM

$$= \frac{1}{4\pi^2} \frac{d^3\vec{P}_C}{2E_C} \frac{d^3\vec{P}_D}{2E_D}\, \delta(\underbrace{E_A + E_B}_{W\,=\,\sqrt{s}} - E_C - E_D)\, \delta^3(-\vec{P}_C - \vec{P}_D)$$

$$= \frac{1}{4\pi^2} \frac{d^3\vec{P}_C}{2E_C\, 2E_D}\, \delta(W - E_C - E_D)$$ $\rightarrow$ we integrated over P_D



$\vec{P}_A = -\vec{P}_B$

P_A : 4-vector
$\vec{P}_A$: 3-vector
P_A : scalar

CM Frame, em scattering (cont)

$$d^3\vec{p}_c = d^3\vec{p}_f = p_f^2 \, dp_f \underbrace{\sin\theta \, d\theta \, d\varphi}_{d\Omega}$$

so we have:

$$dQ = \frac{1}{4\pi^2} \frac{p_f^2 \, d\Omega}{4 E_c E_D} dp_f \, \delta(W - E_3 - E_4)$$

$$W = E_c + E_D = \sqrt{m_c^2 + p_f^2} + \sqrt{m_D^2 + p_f^2}$$

$$\frac{dW}{dp_f} = \frac{p_f}{E_c} + \frac{p_f}{E_D}$$

$$dp_f = \frac{dW}{p_f} \frac{E_c E_D}{(E_c + E_D)}$$

CM Frame, eμ scattering (cont)

we now have:

$$dQ = \frac{1}{4\pi^2}\frac{p_f}{4}\left(\frac{1}{E_C+E_D}\right) dW\, d\Omega\, \delta(W-E_C-E_D)$$

$$= \boxed{\frac{1}{4\pi^2}\frac{p_f}{4\sqrt{s}}\, d\Omega}$$

Now we find F:

From last lecture we had $F = |\vec{v}_A|\, 2E_A\, 2E_B$

$\vec{v}_A = \dfrac{\vec{p}_A}{E_A}$, for a collinear collision

$$F = |\vec{v}_A - \vec{v}_B| \cdot 2E_A \cdot 2E_B \quad , \quad |\vec{v}_A - \vec{v}_B| = (|\vec{v}_A| + |\vec{v}_B|)$$

$$= \left(\frac{|\vec{p}_A|}{E_A} + \frac{|\vec{p}_B|}{E_B}\right) \cdot 2E_A \cdot 2E_B$$

$$= 4(|\vec{p}_A| E_B + |\vec{p}_B| E_A) = \boxed{4p_i (E_A + E_B) = 4p_i\sqrt{s}}$$

CM Frame, eμ scattering (cont)

We then get: $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{64\pi^2} \dfrac{p_F}{p_i} \dfrac{|\mathcal{M}|^2}{s}$

— Now, we evaluate $|\mathcal{M}|^2$ [neglecting masses]

we have $p_A = (p, \vec{p}_i)$, $p_B = (p, -\vec{p}_i)$

$p_C = (p, \vec{p}_F)$, $p_D = (p, -\vec{p}_F)$

$$-i\mathcal{M} = (ie(p_A + p_C)^\mu)\left(\frac{-i g_{\mu\nu}}{q^2}\right)(ie(p_B + p_D)^\nu)$$

$q^2 = (p_D - p_B)^2$

$(p_A + p_C) = (2p, \vec{p}_i + \vec{p}_F)$

$(p_B + p_D) = (2p, -(\vec{p}_i + \vec{p}_F))$

$(p_D - p_B) = (0, -\vec{p}_F + \vec{p}_i)$

CM Frame, eμ scattering (cont)

$$q^2 = (P_D - P_B)^2 = (0, \vec{p}_1 - \vec{p}_f)^2$$

$$= 0 - (\vec{p}_1 - \vec{p}_f)^2$$

$$= -\left[p_f^2 + p_i^2 - 2 p_f p_i \cos\theta \right]$$

$$= -\left[2p^2 - 2p^2 \cos\theta \right] = -(2p^2(1-\cos\theta))$$

$$(P_A + P_C)^\mu (P_B + P_D)_\mu =$$

$$= 4p^2 - \underbrace{\left[(\vec{p}_i + \vec{p}_f) \cdot (-\vec{p}_i - \vec{p}_f) \right]}$$

$$= -p^2 - p^2 - p^2 \cos\theta - p^2 \cos\theta$$

$$= 6p^2 + 2p^2 \cos\theta$$
$$= 2p^2 (3 + \cos\theta)$$

CM Frame, $e\mu$ scattering (cont)

we had $-i\mathcal{M} = \cdots \; ie \cdot -i \cdot ie \; \cdots$

$$\mathcal{M} = \; = i \cdot ie \cdot -i \cdot ie \; \cdots$$
$$= -e^2 \cdots$$

$$\mathcal{M} = \frac{-e^2 \cdot 2p^2(3+\cos\theta)}{-2p^2(1-\cos\theta)}$$

$$= e^2 \frac{(3+\cos\theta)}{(1-\cos\theta)}$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2} \frac{p_f}{p_i} \cdot \frac{1}{s} \cdot \frac{(3+\cos\theta)^2}{(1-\cos\theta)^2}$$

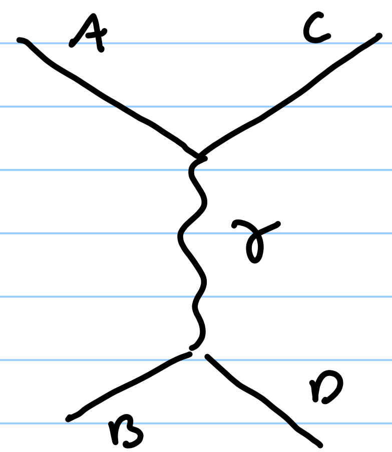
$\alpha = \frac{e^2}{4\pi}$

$\alpha^2 = \frac{e^4}{16\pi^2}$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \cdot \frac{(3+\cos\theta)^2}{(1-\cos\theta)^2}}$$

Spinless electron - electron scattering

we now have 2 diagrams :



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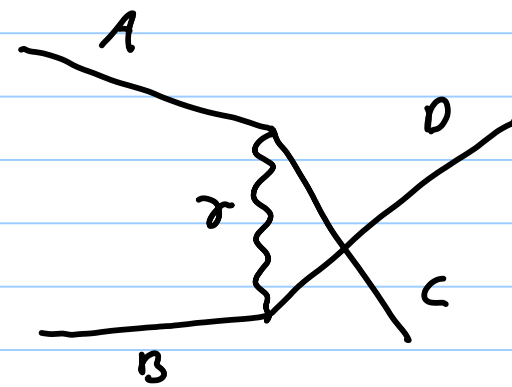
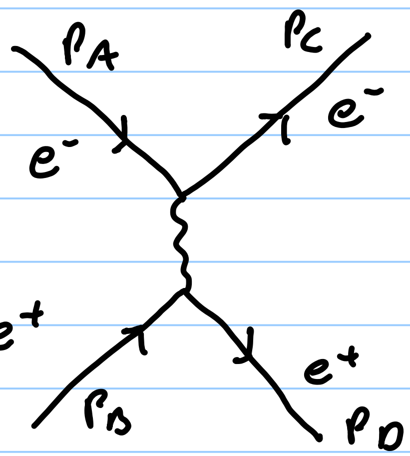


giving us

$$-i\mathcal{M} = -i\left(\underbrace{-\frac{e^2 (p_A + p_C)_\mu (p_B + p_D)^\mu}{(p_D - p_B)^2}}_{\text{what we had for } e\mu \text{ scattering}} - \frac{e^2 (p_A + p_D)_\mu (p_B + p_C)^\mu}{(p_C - p_B)^2} \right)$$

Electron - Positron scattering

we have 2 diagrams :



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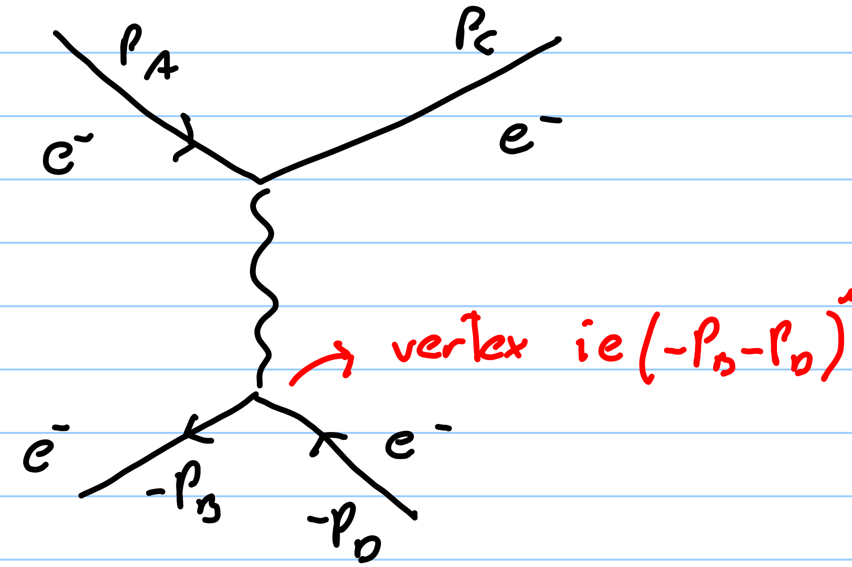

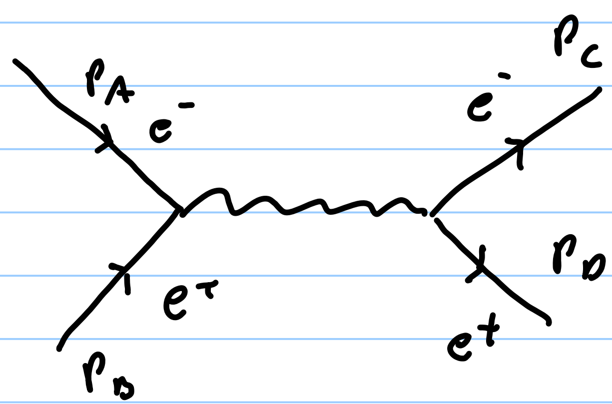

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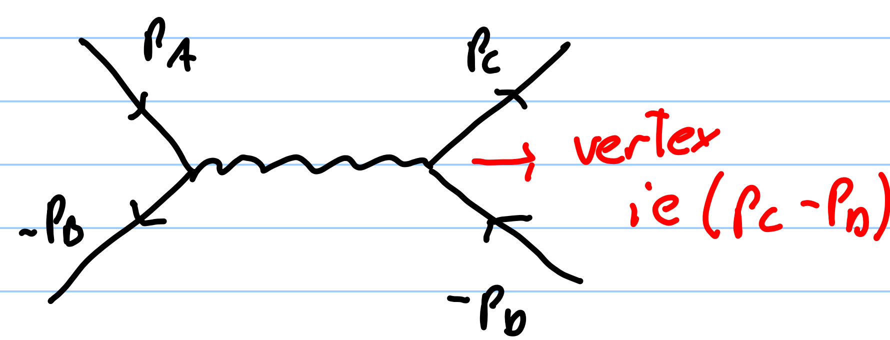

$$-i\mathcal{M}_{e^-e^+} = -i\left[\frac{-e^2(p_A+p_C)_\mu(-p_D-p_B)^\mu}{(p_D-p_B)^2} - \frac{e^2(p_A-p_B)_\mu(-p_D+p_C)^\mu}{(p_C+p_D)^2}\right]$$

Invariant variables (Mandelstam variables)

$$s = (p_A + p_B)^2$$

$$t = (p_A - p_C)^2$$

$$u = (p_A - p_D)^2$$

For $e^- e^+ \to e^- e^+$ in CM we get

$$p_A = (E, \vec{p}_i) \quad , \quad p_B = (E, -\vec{p}_i)$$

$$p_A + p_B = (2E, 0)$$

$$(p_A + p_B)^2 = 4E^2 = 4(p^2 + m^2) = s$$

$\hookrightarrow m_{e^-} = m_{e^+}$

T?

$$p_C = (E, \vec{p}_F)$$

$$(p_A - p_C) = (0, \vec{p}_i - \vec{p}_F)$$

$$(p_A - p_C)^2 = -\left[(\vec{p}_i - \vec{p}_F)(\vec{p}_i - \vec{p}_F) \right]$$

Invariant variables (Mandelstam variables) cont.

$$T = -\left[p_i^2 + p_f^2 - 2\,p_i p_f \cos\theta\right]$$

$$= -\left[2p^2 (1 - \cos\theta)\right] \qquad p_i = p_f \text{ in CM}$$

$$u = -2p^2 (1 + \cos\theta)$$

We can rewrite $\mathcal{M}_{e^-e^-}$ in terms of Mandelstam variables:

$$\mathcal{M}_{e^-e^-} = e^2\left(\frac{u-s}{t} + \frac{t-s}{u}\right)$$

$$\mathcal{M}_{e^+e^-} = e^2\left(\frac{s-u}{t} + \frac{t-u}{s}\right)$$

Decay Rates

$$A \to 1 + 2 + \dots + N$$

$$d\Gamma = \frac{1}{2E_A} |M|^2 \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \cdots \frac{d^3\vec{p}_N}{(2\pi)^3 2E_N} \delta^4(p_A - p_1 - p_2 - \dots - p_N)$$

For particle A decaying particles 1 and 2:

$$\Gamma = \frac{1}{m_A 32\pi^2} \int \frac{|M|^2}{E_1 E_2} \delta^4(p_A - p_1 - p_2)\, d^3\vec{p}_1\, d^3\vec{p}_2$$

- Example: Find Γ for the process $\pi^0 \to \gamma\gamma$ if the invariant amplitude is M.

$$\delta^4(p_A - p_1 - p_2) = \delta(m_A - E_1 - E_2)\, \delta^3(-\vec{p}_1 - \vec{p}_2)$$

$$m_1 = m_2 = 0 \Rightarrow E_1 = p_1 \text{ and } E_2 = p_2$$

$$\to \Gamma = \frac{1}{32\pi^2 m_A} \int \frac{|M|^2}{p_1 p_2} \delta(m_A - p_1 - p_2)\, \delta^3(-\vec{p}_1 - \vec{p}_2)\, d^3\vec{p}_1\, d^3\vec{p}_2$$

Example continued

$$\Gamma = \frac{|M|^2}{32\pi^2 m_A} \int \frac{|M|^2}{p_1^2}\, \delta(m_A - 2p_1)\, d^3\vec{p}_1$$

$$d^3\vec{p}_1 = p_1^2 \sin\theta\, d\theta\, d\varphi \quad , \quad \int \sin\theta\, d\theta\, d\varphi = 4\pi$$

so:

$$\Gamma = \frac{|M|^2}{8\pi m_A} \int_0^\infty \delta(m_A - 2p_1)\, dp_1$$

$$\delta(m_A - 2p_1) = \frac{1}{2}\,\delta\left(p_1 - \frac{m_A}{2}\right)$$

$$\boxed{\Gamma = \frac{1}{16\pi m_A} |M|^2}$$

Cross Section for $e\mu$ scattering

Adding spin → DIRAC EQUATION

START FROM: $$H\psi = (\alpha\cdot\vec{p} + \beta m)\psi \qquad (1)$$

↳ linear in $\frac{\partial}{\partial t}$, ∇

and: $$H^2\psi = (p^2 + m^2)\psi \qquad (2)$$

↳ follows $E^2 = p^2 + m^2$

DIRAC-PAULI REP.

$$\gamma^\mu \equiv (\beta, \beta\alpha) \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Equation (cont)

We want to find eigenvectors for (1)

$$Hu = (\alpha \cdot \vec{p} + \beta m)\, u = E u$$

4 solutions : Two with $E > 0$
Two with $E < 0$

First look at particle at rest:

$p = 0$

$$\rightarrow \quad Hu = \beta m u = \begin{pmatrix} mI & 0 \\ 0 & mI \end{pmatrix} u$$

Eigenvalues: $m, -m$

$E < 0$ solutions are to be intepreted as $E > 0$ solutions for anti-particles

Eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

DIRAC EQUATION (cont.)

We now consider the case where $p \neq 0$

$$H u = \begin{pmatrix} m & \sigma\cdot\vec{p} \\ \sigma\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\sigma\cdot\vec{p}\, u_B = (E-m)\, u_A \qquad (3)$$

$$\sigma\cdot\vec{p}\, u_A = (E+m)\, u_B \qquad (4)$$

First, for $E>0$ solutions, we set $u_A^{(s)} = \chi^{(s)}$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From (4) $\quad u_B^{(s)} = \dfrac{\sigma\cdot\vec{p}}{E+m}\chi^{(s)}$

$$\rightarrow \quad u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \frac{\sigma\cdot p}{E+m}\chi^{(s)} \end{pmatrix} \qquad E>0, \quad s=1,2$$

$$N = \sqrt{E+m}$$

Dirac Equation (cont.)

we get for $E<0$, $s=3,4$

$$v^{(s)} = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

Completeness relations

$$\sum_{s=1,2} u^{(s)}(p)\,\bar{u}^{(s)}(p) = \not{p}+m$$

$$\sum_{s=1,2} v^{(s)}(p)\,\bar{v}^{(s)}(p) = \not{p}-m$$

↳ positron

From (1) we get

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0 \qquad (3)$$

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \qquad (4)$$

Hermitian conjugate

Some useful relations: $\bar{\psi} \equiv \psi^\dagger\gamma^0$

$\gamma^{0\dagger} = \gamma^0$, $(\gamma^0)^2 = I$, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

$\gamma^{k\dagger} = -\gamma^k$ $\qquad$ $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

$\not{p} \equiv \gamma^\mu p_\mu$

Dirac Equation (cont.)

Other results: $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$, $\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad \gamma^{5\dagger} = \gamma^5 , \; (\gamma^5)^2 = I$$
$$\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$$

We now find an expression for the current density:

TAKE (3) $\times \bar{\psi}$ FROM THE LEFT AND
TAKE (4) $\times \psi$ FROM THE RIGHT:

$$\bar{\psi}\gamma^\mu\partial_\mu\psi + (\partial_\mu\bar{\psi})\gamma^\mu\psi = \partial_\mu \underbrace{(\bar{\psi}\gamma^\mu\psi)}_{j^\mu} = 0$$

in fact we need to identify j^μ with the charge current density.

$$\boxed{j^\mu = -e\bar{\psi}\gamma^\mu\psi}$$

Cross Section Calculations (adding spin)

Electron with 4-momentum p^μ:

$$\psi = u(\vec{p})\, e^{-ip\cdot x} \qquad \text{will satisfy} \quad (\gamma_\mu p^\mu - m)\psi = 0$$

we substitute: $p^\mu \to p^\mu + eA^\mu$

$$(\gamma_\mu p^\mu - m)\psi = \underbrace{\gamma^0 V}_{= -e\gamma_\mu A^\mu}\psi$$

We go back to our transition amplitude:

$$T_{fi} = -i\int \psi_f^\dagger(x)\, V(x)\, \psi_i(x)\, d^4x$$

$$= ie\int \bar{\psi}_f \gamma_\mu A^\mu \psi_i\, d^4x \quad \rightarrow \quad \boxed{\begin{aligned} j^\mu &= -e\bar{\psi}_f \gamma^\mu \psi_i \\ j^\mu &= -e\bar{u}_f \gamma_\mu u_i\, e^{i(p_f - p_i)\cdot x} \end{aligned}}$$

$$= -i\int j_\mu^{fi} A^\mu\, d^4x$$

Note: now vertex factor has 4x4

Note: now interaction involves magnetic moment of electron