LECTURE 6: Spontaneous Symmetry Breaking (Part II)

Overview:

- Recap of Abelian case
- Ginzburg-Laudau
- Non-Abelian gauge theories (contruction)
- Higgs Mechanism (non-Abelian case)

(This lecture mostly follows Quigg Chapters 4-5)

Higgs Mechanism (Abelian case recap)

We saw that spontaneous breaking of a continuous symmetry leads to massless bosons (Goldstone bosons). We expect one massless boson per broken generator.

We saw however that in the case of a local gauge theory, the massless gauge boson and the massless Goldstone boson conspire to give us a massive gauge boson without the massless Goldstone boson.

In the case we studied, we had before symmetry breaking:

2 scalars: 2 degrees of freedom
1 massless vector boson: 2 degrees of freedom

Total =4

After breaking we had (explicit in unitary gauge):

1 massive vector boson: 3 degrees of freedom
1 massive Higgs scalar: 1 degree of freedom

Total =4

Ginzburg Landau Superconductivity

Ψ : macroscopic wave function describing condensate

Free energy of superconductor can be written as:

$$G_{super}(0) = G_{normal}(0) + \alpha|\Psi|^2 + \beta|\Psi|^4$$

$$G_{super}(B) = G_{super}(0) + \frac{B^2}{2} + \frac{1}{2m^*}\Psi^*(-i\nabla - e^*A|^2\Psi$$

in weak field approx., field equations derived using $G_{super}(B)$ lead to massive photon

Meisner Effect:

- **Cooper pairs form BEC condensate below $T_c \sim 10^0$-10^2 K. Condensate disturbed by EM field**
- **Short range force, attenuation length ~10^{-6}cm**
- **equivalent to photon acquiring a mass**

Electroweak symmetry breaking:

- **Higgs condenses below $T_c \sim 10^{15}$K. Condensate disturbed by gauge bosons**
- **Short range force, attenuation length ~10^{-18}cm**
- **W/Z bosons acquire mass**

Non-Abelian gauge theories

For the Abelian case we studied, $U(1)$, we saw that imposing local gauge transformations:

$\varphi(x) \to e^{i\alpha(x)}\varphi(x)$ required addition of gauge-covariant derivative to keep theory invariant under those transformations: $D_\mu \equiv \partial_\mu + iq A_\mu(x)$

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha$$

We now look at an example of a non-Abelian gauge theory: $SU(2)$-isospin gauge theory

$$\psi = \begin{pmatrix} \text{proton} \\ \text{neutron} \end{pmatrix}$$

Global symmetry $\to$ freedom to choose what we call a proton and neutron everywhere

Non-Abelian gauge theories

Consider a local gauge Transformation for the Field $\psi(x)$:

$$\psi(x) \rightarrow \psi(x)' = G(x)\psi(x)$$

with $G(x) \equiv \exp\left(\frac{i}{2}\tau \cdot \alpha(x)\right)$

τ are the Pauli matrices

$\alpha \equiv \alpha_1, \alpha_2, \alpha_3$

$$\partial_\mu \psi \rightarrow G(\partial_\mu \psi) + \underline{(\partial_\mu G)}\psi$$

To Take care of This Term, let's introduce a gauge covariant derivative:

$$D_\mu \equiv I\partial_\mu + igB_\mu$$

with $\underline{B_\mu} = \frac{1}{2}\tau^a b_\mu^a = \frac{1}{2}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

2x2 matrix

$b_\mu = (b_1, b_2, b_3)$

Non-Abelian gauge theories (cont.) ⑥

How does B_μ need to transform to cancel extra terms?

We want $\quad D_\mu \psi \to D_\mu' \psi' = G(D_\mu \psi)$

$$D_\mu' \psi' = (\partial_\mu + ig B_\mu') \psi'$$
$$= G(\partial_\mu \psi) + (\partial_\mu G)\psi + ig D_\mu'(G\psi)$$
$$\stackrel{!}{=} G(\partial_\mu + ig B_\mu)\psi$$
$$= G(\partial_\mu \psi) + ig\, G(B_\mu \psi)$$

$$\Rightarrow \quad ig B_\mu'(G\psi) = ig\, G(B_\mu \psi) - (\partial_\mu G)\psi$$

multiply from the right by G^{-1}

$$\rightarrow \quad B_\mu' = G B_\mu G^{-1} + \frac{i}{g}(\partial_\mu G) G^{-1}$$
$$= G\left[B_\mu + ig\, G^{-1}(\partial_\mu G)\right] G^{-1}$$

Non-Abelian gauge theories (cont.) ⑦

$$B_\mu' = G\left[B_\mu + \frac{i}{g} G^{-1} (\partial_\mu G)\right] G^{-1} \qquad (1)$$

Looks complicated... let's try it with $G_{em} = e^{iq\alpha(x)}$

We get: $$A_\mu' = e^{iq\alpha(x)}\left[A_\mu + \frac{i}{q} \cdot e^{-iq\alpha(x)} \cdot iq\,\partial_\mu \alpha\, e^{iq\alpha(x)}\right] e^{-iq\alpha(x)}$$

$$= A_\mu - \partial_\mu \alpha \qquad \text{(ok it works...)}$$

Consider infinitesimal gauge transformation $(|\alpha_i| \ll 1)$

$G = 1 + \frac{i}{2}\tau \cdot \alpha$ using (1) above we get:

$$G B_\mu G^{-1} = \left[B_\mu + \frac{i}{2}\tau \cdot \alpha B_\mu\right] \cdot 1 - \frac{i}{2}\tau \cdot \alpha$$

$$= B_\mu + \overbrace{\frac{i}{2}\tau \cdot \alpha\, B_\mu - \frac{i}{2} B_\mu\, \tau \cdot \alpha} + O(\alpha^2)$$

$$\frac{i}{g}(\partial_\mu G) G^{-1} = \left[\frac{i}{g}\, \frac{i}{2}\, \partial_\mu (\alpha \cdot \tau)\right] \cdot \left(1 - \frac{i}{2}\tau \cdot \alpha\right) = -\frac{1}{2g}\partial_\mu (\alpha \cdot \tau)$$

Non-Abelian gauge theories (cont.)

we have

$$B'_\mu = B_\mu + \frac{i}{2}\alpha\cdot\tau B_\mu - \frac{i}{2}B_\mu \alpha\cdot\tau - \frac{1}{2g}\partial_\mu(\alpha\cdot\tau) + \ldots$$

now $B_\mu = \frac{1}{2}\tau\cdot b_\mu$ so we get:

$$\tau\cdot b'_\mu = \tau\cdot b_\mu + \frac{i}{2}\left(\tau\cdot\alpha\,\tau\cdot b_\mu - \tau\cdot b_\mu\,\alpha\cdot\tau\right) - \frac{1}{g}\partial_\mu(\alpha\cdot\tau)$$

in component form: $\frac{i}{2}\alpha^j b^k_\mu(\tau^j\tau^k - \tau^k\tau^j) = \frac{i}{2}\alpha^j b^k_\mu[\tau^j,\tau^k]$

For $SU(2)$: $[\tau^j,\tau^k] = 2i\varepsilon_{jkl}\tau^l$

second term becomes: $-\varepsilon_{jkl}\alpha^j b^k_\mu \tau^l = -\alpha\times b_\mu\cdot\tau$

isospin components are linearly indep. so:

$\tau\cdot b'_\mu = \tau\cdot b_\mu - \alpha\times b_\mu\cdot\tau - \frac{1}{g}\partial_\mu(\alpha\cdot\tau)$ becomes:

$$b'^l_\mu = b^l_\mu - \underbrace{\varepsilon_{jkl}\alpha^j b^k_\mu} - \frac{1}{g}\partial_\mu\alpha^l$$

Non-Abelian gauge theories (cont.)

Note: new Term $[\varepsilon_{jkl}\,\partial^j b_\mu^k]$ is picked because gauge Transformations do not commute

Now we focus on field strength Tensor. We would like it To Transform as:

$$F_{\mu\nu}' = G F_{\mu\nu} G^{-1}$$

if we Try

$\partial_\nu B_\mu' - \partial_\mu B_\nu'$ for $F_{\mu\nu}'$

we do not get

$$G(\partial_\nu B_\mu - \partial_\mu B_\nu) G^{-1}$$

$F_{\mu\nu} \rightarrow F_{\mu\nu}^a \tau^a$

$\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu})$

and $Tr(\tau^a \tau^b) = 2\delta^{ab}$

Are we missing a commutator again?

Non-Abelian gauge theories (cont.)

if we try $F_{\mu\nu} = \frac{1}{iq}[D_\nu, D_\mu]$ for QED:

$$F_{\mu\nu} = \frac{1}{iq}[(\partial_\nu + iqA_\nu), (\partial_\mu + iqA_\mu)]$$

$$= \partial_\nu A_\mu - \partial_\mu A_\nu + iq\underbrace{[A_\nu, A_\mu]}_{=0}$$

$$F_{\mu\nu} = \frac{1}{ig}[D_\nu, D_\mu] = \partial_\nu B_\mu - \partial_\mu B_\nu + ig[B_\nu, B_\mu]$$

$$G F_{\mu\nu} G^{-1} = F'_{\mu\nu}$$

Yang-Mills Lagrangian: $\bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$

in component form: $F^\ell_{\mu\nu} = \partial_\nu B^\ell_\mu - \partial_\mu B^\ell_\nu + g\,\varepsilon_{jk\ell}\, b^j_\mu b^k_\nu$

$$b'^{\ell}_\mu = b^\ell_\mu - \frac{1}{g}\partial_\mu \alpha^\ell - \varepsilon_{jk\ell}\,\alpha^j b^k_\mu$$

For other groups, we'll replace $\varepsilon_{jk\ell}$ by the group's structure constants $F_{jk\ell}$

Non-Abelian gauge theories (cont.)

$$g_1 = 1 + i w_1^\alpha T_\alpha \qquad g_2 = 1 + i w_2^\alpha T_\alpha , \quad g_1 g_2 = 1 + i (w_1^\alpha + w_2^\alpha) T_\alpha + \dots$$

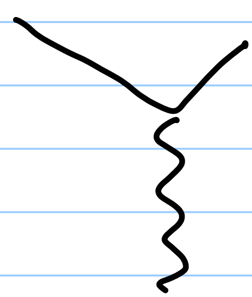
$$g_1 g_2 (g_2 g_1) \approx (1 + i w_1^\alpha T_\alpha)(1 + i w_2^\beta T_\beta)(1 - i w_2^\beta T_\beta)(1 - i w_1^\alpha T_\alpha)$$
$$= 1 - w_1^\alpha w_2^\beta [T_\alpha, T_\beta] \quad , \quad [T_\alpha, T_\beta] = i f^\gamma_{\alpha\beta} T_\gamma$$

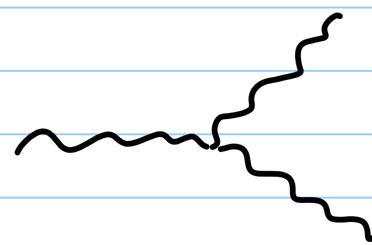
see Appendix B of Burgess and Moore

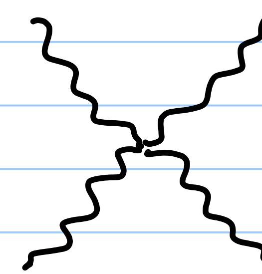
Also note that the non-Abelian theory has additional couplings:

$$F \propto (\partial B - \partial B) + g B B$$

$$\mathcal{L} \propto \underbrace{(\partial B - \partial B)}_{\text{propagator}} + g \underbrace{(\partial B - \partial B) B B}_{\text{triple}} + \underbrace{g^2 B B B B}_{\text{quartic}}$$







Higgs Mechanism (non-Abelian case)

We will study an SU(2) doublet of complex scalar fields:

$$\varphi = \sqrt{\tfrac{1}{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}$$

The Lagrangian is: $(\partial_\mu \varphi)^\dagger (\partial^\mu \varphi) - \mu^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2$

The covariant derivative: $D_\mu = \partial_\mu + ig\frac{\tau_a}{2} B_\mu^a$

Under infinitesimal transformation: $\varphi(x)' = (1 + \frac{i}{2}\alpha(x)\cdot\tau)\,\varphi$

$$B_\mu' = B_\mu - \frac{1}{g}\partial_\mu \alpha - \alpha \times B_\mu$$

We obtain

$$\mathcal{L} = (\partial_\mu \varphi + ig\frac{\tau}{2}\cdot B_\mu \varphi)^\dagger (\partial^\mu \varphi + ig\frac{\tau}{2}\cdot B^\mu \varphi) - V(\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$V(\varphi) = \mu^2 (\varphi^\dagger \varphi) + \lambda (\varphi^\dagger \varphi)^2$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - g B_\mu \times B_\nu$$

Higgs Mechanism (non-Abelian case)

minimum of potential at $\varphi^\dagger\varphi = \frac{1}{2}\left(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2\right) = \frac{-\mu^2}{2\lambda}$

we chose minimum around which to do our expansion: $\varphi_3^2 = -\frac{\mu^2}{\lambda} \equiv v^2$ $\quad \varphi_1=\varphi_2=\varphi_4=0$

We parametrize fluctuations from the vacuum $\varphi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$ in terms of 4 real scalar fields $\xi_1, \xi_2, \xi_3, \eta$

$$\varphi(x) = e^{i\tau\cdot\xi(x)/v}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v+\eta\end{pmatrix} \approx \frac{1}{\sqrt{2}}\begin{pmatrix}1+i\xi_3/v & i(\xi_1-\xi_2)/v\\ i(\xi_1+\xi_2)/v & 1-i\xi_3/v\end{pmatrix}\begin{pmatrix}0\\v+\eta\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_2+i\xi_1\\ v+\eta-i\xi_3\end{pmatrix}, \quad \text{so} \quad \varphi^\dagger\varphi =$$

$$\frac{1}{\sqrt{2}}\left(\xi_2-i\xi_1,\ v+\eta+i\xi_3\right)\frac{1}{\sqrt{2}}\begin{pmatrix}\xi_2+i\xi_1\\ v+\eta-i\xi_3\end{pmatrix} = \xi_1^2+\xi_2^2+\xi_3^2+v^2+\eta^2+2v\eta$$

We know all terms from V will cancel save $\lambda\mu^2\eta$ ↳ massive scalar

→ for small oscillations

Higgs Mechanism (non-Abelian case)

Let's move to the unitary gauge right away:

$$\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+\eta \end{pmatrix}$$

$$D_\mu \varphi = \frac{1}{\sqrt{2}} \left(\partial_\mu \begin{pmatrix} 0 \\ \eta \end{pmatrix} + \frac{ig}{2} \begin{pmatrix} b^3_\mu & \sqrt{2}\, b^-_\mu \\ \sqrt{2}\, b^+_\mu & -b^3_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v+\eta \end{pmatrix} \right)$$

where $b^\pm_\mu = \frac{1}{\sqrt{2}} (b^1_\mu \pm i b^2_\mu)$

$$|D_\mu \varphi|^2 = \frac{1}{2} \partial_\mu \eta\, \partial^\mu \eta + \frac{1}{4} g^2 v^2 \left(b^+_\mu b^{\mu -} + \frac{1}{2} b^3_\mu b^{3\mu} \right)$$

$$+ \frac{1}{4} g^2 \eta^2 \left(b^+_\mu b^{-\mu} + \frac{1}{2} b^3_\mu b^{\mu 3} \right) + \frac{1}{2} g^2 v \eta \left(b^+_\mu b^{\mu -} + \frac{1}{2} b^3_\mu b^{\mu 3} \right)$$

$\rightarrow$ 3 bosons with mass $\frac{gv}{2}$

Higgs Mechanism (non-Abelian case)

Summary:

we started with: 4 scalars : 4 dof
3 massless bosons: $3 \times 2 = 6$ dof

$\boxed{\text{Total} = 10 \text{ dof}}$

we end up with : 1 scalar (massive) : 1 dof
3 massive bosons : $3 \times 3 = 9$ dof

$\boxed{\text{Total} = 10 \text{ dof}}$

In the Standard Model we have 3 massive vector bosons (2 charged, one neutral) and one massless boson (neutral).

→ see problem set #1

Problem set 1

Due Friday 15^{th} 5pm
in my mailbox in MP 804
(or give To Winnie Kan)

Problem 1

Analyse the spontaneous breaking of a global SU(2) symmetry for the case of 3 real scalar fields in an SU(2) Triplet: $\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)\cdot(\partial^\mu \varphi) - V(\varphi)$$

$$V(\varphi) = \frac{1}{2}\mu^2 \varphi\cdot\varphi + \frac{1}{4}|\lambda|(\varphi\cdot\varphi)^2$$

Problem 2

Analyse the spontaneous breaking of a local SU(2) symmetry for the case of 3 real scalar fields in an SU(2) Triplet: $\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$

→ use $\mathcal{L}$ and V from page 12

Problem set # 1 (cont)

Problem 3:

We now Turn To the Following $SU(2) \times U(1)$ Lagrangian:

$$\mathcal{L} = \left| \left(i\partial_\mu - g\frac{\tau}{2}\cdot W_\mu - \frac{g'}{2} B_\mu \right) \varphi \right|^2 - V(\varphi)$$

We use the Followin doublet of complex scaler Fields

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \qquad \varphi^+ \equiv (\varphi_1 + i\varphi_2)/\sqrt{2}$$

$$\varphi^0 = (\varphi_3 + i\varphi_4)/\sqrt{2}$$

Obtain the mass of the vector bosons using the relevent Term: $\left| \left(-ig\frac{\tau}{2}\cdot W_\mu - i\frac{g'}{2} B_\mu \right) \varphi \right|^2$

you will get an off-diagonal Term for W_μ^3 and B_μ

Express your result in Terms of physical Fields that diagonalize the mass matrix.

you can use $\frac{g'}{g} = \tan\theta_W$