# LECTURE 7: Weak Interactions (Part I)

Overview:

- A closer look at SU(2) (and other groups)
- A little group theory
- Inverse muon decay

(This lecture mostly follows Griffiths Chapter 10)

# U(1) and SU(2) recap



U(1) : $\varphi \rightarrow \varphi' = \exp(i\alpha(x))\,\varphi$

SU(2) $\varphi \rightarrow \varphi' = \exp\left(\frac{i}{2}\underbrace{\tau}_{\text{3 Pauli Matrices}} \cdot \underbrace{\alpha(x)}_{\alpha_1,\alpha_2,\alpha_3}\right)\varphi$

Both cases: $\partial_\mu \varphi \rightarrow G(\partial_\mu \varphi) + \underbrace{(\partial_\mu G)\varphi}$

To take care of extra term introduce gauge covariant derivative:

U(1) : $D_\mu = \partial_\mu + ieA_\mu$

SU(2): $D_\mu = I\partial_\mu + igB_\mu$ $\quad B_\mu = \frac{1}{2}\tau \cdot \underbrace{b_\mu}_{b_\mu^1,\, b_\mu^2,\, b_\mu^3}$

Both cases: $C_\mu' = G C_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}$

U(1) : $A_\mu \rightarrow A_\mu' = A_\mu - \frac{1}{e}\partial_\mu \alpha$

SU(2) : $b_\mu \rightarrow b_\mu' = b_\mu - \alpha \times b_\mu - \frac{1}{g}\partial_\mu \alpha$

or $b_\mu^{\ell\,\prime} = b_\mu^\ell - \varepsilon_{jk\ell}\,\alpha^j b^k - \frac{1}{g}\partial_\mu \alpha^\ell$

# Group Theory Interlude I (see Aitchison and Hey)



A group is a set of elements $(a, b, c, \ldots)$ with a law to combine any two elements to form their ordered product $ab$ such that:

1- For every $a, b \in G$, the product $ab \in G$
2- $(ab)c = a(bc)$
3- $G$ has a unique element $e$ such that for all $a \in G$: $ae = ea = a$
4- For all $a \in G$, there is a unique element $a^{-1}$ such that $a^{-1}a = aa^{-1} = e$

if $ab = ba$ : Abelian group
if $ab \neq ba$ : non-Abelian group

examples: following set forms Abelian group $(1, i, -1, -i)$

following matrices too:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

# Group Theory Interlude (Lie Groups)

We are interested in continuous groups with elements labelled by continuously variable real parameters

$$\alpha_1, \alpha_2, \alpha_3, \dots \alpha_r \quad , \quad g(\alpha_1, \alpha_2, \alpha_3, \dots \alpha_r) \equiv g(\alpha)$$

For a continuous group, condition 1 takes the form:

$$g(\alpha)\, g(\beta) = g(\gamma(\alpha, \beta))$$

↳ continuous functions of $\alpha, \beta$

→ if $\gamma$ is an analytic function, then we have a Lie Group

↳ implies that it can be expressed as a power series. We can use a power series expansion to move from one element to another (within its neighborhood).

Lie proved that the properties of the elements which can be reached continuously from $I$ are determined from elements in the neighbourhood of $I$

# Group theory Interlude (Lie Groups II)

Consider a group of transformations defined by:

$$x'_i = f_i(x_1, x_2, \dots x_N; \alpha_1, \alpha_2, \dots, \alpha_r)$$

$x_i$ : "coordinates" → or field components on which the transformations act.

$\alpha_j$ : parameters of the transformations

$\alpha = 0$ is the identity transformation $\Rightarrow x_i = f_i(x, 0)$

$$dx_i = \sum_{\nu=1}^{r} \frac{\partial f_i}{\partial \alpha_\nu} d\alpha_\nu \qquad \text{(partial deriv. evaluated at } (x,0))$$

$$F \rightarrow F + dF = F + \sum_{i=1}^{N} \frac{\partial F}{\partial x_i} dx_i = F + \sum_{i=1}^{N} \left[ \sum_{\nu=1}^{r} \frac{\partial f_i}{\partial \alpha_\nu} d\alpha_\nu \right] \frac{\partial F}{\partial x_i}$$

$$\equiv \left[ 1 - \sum_{\nu=1}^{r} d\alpha_\nu \, i\hat{X}_\nu \right] F \qquad \text{with:}$$

$$\boxed{\hat{X}_\nu = i \sum_{i=1}^{N} \frac{\partial f_i}{\partial \alpha_\nu} \frac{\partial}{\partial x_i}} \rightarrow \text{generator of infinitesimal transfo.}$$

# Group theory Interlude (Lie Groups III)

For finite Transformations: $\left[1 - \sum_{v=1}^{r} d\alpha_v \, i\hat{X}_v\right]$

becomes $\exp\left[-i\,\alpha\cdot\hat{X}\right]$ $\qquad \alpha\cdot\hat{X} \equiv \sum_v \alpha_v \hat{X}_v$

Theorem states that commutator of any Two generators is a linear combination of the generators:

$$[\hat{X}_\alpha, \hat{X}_\beta] = \underbrace{c^\gamma_{\alpha\beta}}\,\hat{X}_\gamma$$

→ structure constants of the group
commutation relations called
algebra of the group.

Example 1: $SU(2)$

Here the Transformations act on complex Two-component column vector) $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

# Group Theory Interlude (Lie Groups IV)



For infinitesimal $SU(2)$ Transfo.:

$$\begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \left(1 + i\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\tau}}{2}\right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\boxed{\frac{i}{2}\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}}$$

$$dq_1 = \frac{i}{2}\alpha_3 q_1 + \left(\frac{i\alpha_1}{2} + \frac{\alpha_2}{2}\right) q_2$$

$$dq_2 = \frac{-i\alpha_3}{2} q_2 + \left(\frac{i\alpha_1}{2} - \frac{\alpha_2}{2}\right) q_1$$

$$\frac{\partial f_1}{\partial \alpha_1} = \frac{iq_2}{2}, \quad \frac{\partial f_1}{\partial \alpha_2} = \frac{q_2}{2}, \quad \frac{\partial f_1}{\partial \alpha_3} = \frac{iq_1}{2}$$

$$\frac{\partial f_2}{\partial \alpha_1} = \frac{iq_1}{2}, \quad \frac{\partial f_2}{\partial \alpha_2} = -\frac{q_1}{2}, \quad \frac{\partial f_2}{\partial \alpha_3} = \frac{-q_2}{2}$$

$$\hat{X}_1 = \frac{-1}{2}\left\{ q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} \right\}, \quad \hat{X}_2 = \frac{i}{2}\left\{ q_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_2} \right\}$$

$$\hat{X}_3 = \frac{1}{2}\left\{ -q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} \right\}$$

$$\boxed{[\hat{X}_i, \hat{X}_j] = i\varepsilon_{ijk}\hat{X}_k}$$

Example 2 : SU(3)

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}' = \left(1 + \frac{i}{2}\eta \cdot \lambda\right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

now we have 8 parameters, and $\lambda$ are the Gell-Mann matrices e.g. $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

$$\lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

One can determine the generators as in example 2.

SU(3) algebra : $[\hat{G}_a, \hat{G}_b] = i f_{abc} \hat{G}_c$

$f_{123} = 1$, $f_{147} = 1/2$, $f_{458} = \frac{\sqrt{3}}{2}$ etc.

More when we study QCD

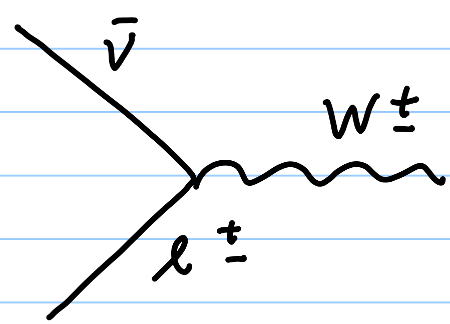
# Group theory Interlude (Lie Groups VI)

We've see particular matrix representations of SU(2) and SU(3). One can use other representations: we just need to respect the group's element multiplication. For SU(2), the triplet representation, which will act on a triplet of fields, must respect the commutation relations. For example, one can write the matrices using $(T^j)_{kl} = -i\varepsilon_{jkl}$

You can check that the 3x3 matrices satisfy the SU(2) commutation relations.

# Weak Interactions (inverse muon decay)

We will now consider simple processes that involve the weak charged current and leptons



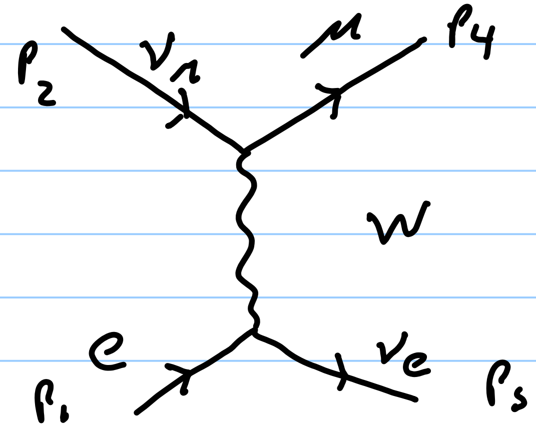

$\rightarrow$ vertex factor $\dfrac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$

propagator: $\dfrac{-i\left(g_{\mu\nu} - q_\mu q_\nu / M_w^2\right)}{q^2 - M_w^2}$

$\hookrightarrow$ simplifies to $\dfrac{i g_{\mu\nu}}{M_w^2}$ if $q \ll M_w$

Consider the reaction

$\nu_\mu + e^- \rightarrow \mu^- + \nu_e$




$$\mathcal{M} = \frac{g_w^2}{8M_w^2}\left[\bar{u}(3)\gamma^\mu(1-\gamma_5)u(1)\right] \cdot \left[\bar{u}(4)\gamma^\mu(1-\gamma_5)u(2)\right]$$

# Weak Interactions (inverse muon decay)

As before we use the relation:

$$\sum_{spins} [\bar{u}(3)\gamma_\mu(1-\gamma_5)u(1)][\bar{u}(3)\gamma_\nu(1-\gamma_5)u(1)]^*$$
$$= Tr[\gamma_\mu(1-\gamma_5)(\not{p}_1+m_1)\gamma_\nu(1-\gamma_5)(\not{p}_3+m_3)]$$

$$\sum_{spins}|M|^2 = \left(\frac{g_w^2}{8M_w^2}\right) Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma^5)\not{p}_3] \times$$
$$Tr[\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma_5)(\not{p}_4+m_\mu)]$$

Trace theorems with $\gamma_5$:

$$Tr(\gamma_5) = 0 \quad , \quad Tr(\gamma^5 \not{a}\not{b}) = 0$$
$$Tr(\gamma_5\gamma_\mu\gamma_\nu) = 0 \quad ; \quad Tr(\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma) = 4i\epsilon_{\mu\nu\lambda\sigma}$$
$$Tr(\gamma_5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon_{\mu\nu\lambda\sigma}a^\mu b^\nu c^\lambda d^\sigma \quad , \quad Tr(\gamma_5\gamma_\mu\gamma_\nu\gamma_\sigma) = 0$$

## Weak Interactions (inverse muon decay)



Terms from First Trace

1. $\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 \quad = 4\,( p_1^\mu p_3^\nu - g^{\mu\nu}(p_1\cdot p_3) + p_1^\nu p_3^\mu$

2. $\gamma^\mu \not{p}_1 \gamma^\nu \cdot -\gamma^5 \not{p}_3 \quad = 4\,i\varepsilon^{\mu\nu\lambda\sigma}\, p_{1\lambda} p_{3\sigma}$

3. $\gamma^\mu(-\gamma^5)\, \not{p}_1 \gamma^\nu \not{p}_3 \quad =$ 2 $\rightarrow$ move $\gamma^5$ by 2 positions

4. $\gamma^\mu(-\gamma^5)\, \not{p}_1 \gamma^\nu (-\gamma^5)\, \not{p}_3 \quad =$ 1

$$= \gamma^5 \gamma^\mu \not{p}_1 \gamma^\nu (-\gamma^5) \not{p}_3$$

$$= \gamma^5\gamma^5\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 = \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3$$

1st Trace $= 8\left[ p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1\cdot p_3) - i\varepsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma} \right]$

2nd Trace $= 8\left[ p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu}(p_2\cdot p_4) - i\varepsilon_{\mu\nu\kappa\tau} p_2^\kappa p_4^\tau \right]$

note that: $\varepsilon^{\mu\nu\lambda\sigma}\varepsilon_{\mu\nu\kappa\tau} = -2\left(\delta^\lambda_\kappa \delta^\sigma_\tau - \delta^\lambda_\tau \delta^\sigma_\kappa\right)$

# Weak Interactions (inverse muon decay)

Multiplying the two trace results we get:

$$
\begin{aligned}
= \Big[ & (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)\cancel{(p_2 \cdot p_3)} - (p_1 \cdot p_3)\cancel{(p_2 \cdot p_4)} \\
& + (p_1 \cdot p_4)\cancel{(p_3 \cdot p_2)} + (p_1 \cdot p_2)(p_3 \cdot p_4) - \cancel{(p_1 \cdot p_3)}(p_2 \cdot p_4) \\
& - (p_2 \cdot p_4)\cancel{(p_1 \cdot p_3)} - (p_2 \cdot p_4)\cancel{(p_1 \cdot p_3)} + 4\,\cancel{(p_1 \cdot p_3)}(p_2 \cdot p_4) \\
& + 2(p_1 \cdot p_2)(p_3\, p_4) - 2(p_1\, p_4)\cancel{(p_3\, p_2)}
\end{aligned}
$$

$$\Rightarrow \sum_{\text{spins}} |M|^2 = 4\left(\frac{g_W}{M_W}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

in CM frame, neglecting electron mass and muon mass

$$p_1 = (p_i, 0, 0, \vec{p_i}), \quad p_2 = (p_i, 0, 0, -\vec{p_i})$$

$$p_3 = (p_F, 0, 0, \vec{p_F}), \quad p_4 = (p_F, 0, 0, -p_F)$$

$$p_1 \cdot p_2 = 2p_i^2, \quad p_3 \cdot p_4 = 2p_F^2$$

# Weak Interactions (inverse muon decay)

$$2p_i = 2p_f = 2E$$

$$\Rightarrow \sum_{\text{spins}} |M|^2 = 4 \cdot 4E^4 \left(\frac{g_w}{M_w}\right)^4 \qquad \to \quad 4E^2 = s$$

$$G_F = \frac{\sqrt{2}}{8} \frac{g_w^2}{M_w^2} \quad , \quad G_F^2 = \frac{1}{32} \frac{g_w^4}{M_w^4}$$

$$\sum_{\text{spins}} |M|^2 = 32\, G_F^2 \cdot s^2$$

$\sum_{\text{spins}}$ → neutrino one spin state
→ electron two spin states

average over initial spins adds $\frac{1}{2}$ factor

$$\langle |M|^2 \rangle = 16\, G_F^2 s^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \overline{|M|}^2 = \frac{1}{4\pi^2} G_F^2 s \quad , \quad \sigma = \frac{G_F^2 s}{\pi}$$