REVIEW PART 1: Calculation of QED Cross Sections and Decay Rates

Overview:

-Experimental considerations

-Transition rates

-Spinless electron-muon scattering

-Cross section and decay rate calculations

(Mostly follows Halzen and Martin. I recommend Burgess and Moore for a more thorough and rigourous treatment)

What do we measure and what do we need to calculate?

-We want to determine various properties of particles and their interactions.

-A common technique is to collide particles at a very high rate and at very high energies. We then look at the products of these collisions. A few notes:

-What we end up observing are the long-lived decay products of what was initially produced in the collision

-There can be a lot of decay products and we need to determine how to relate these to what we initially produced

-The probablility of producing what we are interested in

What do we want to calculate? It depends on what we can measure...

-we want to determine a quantity related to the probablility of producing certain final states that is independent of the rate of the collisions. This quantity should not involve time.

Let's start from here:

Particle in a box of volume V

Free - particle equation: $H \varphi_n = E_n \varphi_n$

$$\int_V \varphi_m^* \varphi_n \, d^3x = \delta_{mn}$$

equation with interaction potential:

$$(H + V(x,t))\psi = i \frac{\partial \psi}{\partial t}$$

Solutions to $(H + V(x,t))\Psi = i\frac{\partial \Psi}{\partial t}$ can be expressed as $\Psi = \sum_n a_n(t)\,\varphi_n(x)\,e^{-iE_nt}$ (1)

- want to find the $a_n(t)$

insert (1) into Schrödinger equation:

$$\sum_n E_n a_n \cancel{\varphi_n(x)}\, e^{-iE_nt} + \sum_n V(x,t)\, a_n \varphi_n(x)\, e^{-iE_nt}$$

$$= i\sum \frac{da_n}{dt}\varphi_n(x)\, e^{-iE_nt} + i\sum a_n \cancel{\varphi_n(x)} \cdot -iE_n\, e^{-iE_nt}$$

- multiply both sides by φ_f^*, integrate over V

(2) LHS : $i\sum_n \int \frac{da_n}{dT}\varphi_f^*\varphi_n\, e^{-iE_nT}\, d^3x = i\cancel{\sum_n} \frac{da_f}{dT}\, e^{-iE_fT}$

(3) RHS : $\sum_n a_n(t)\int \varphi_f^* V \varphi_n\, d^3x\; e^{-iE_nT}$

From 2 and 3, we get

$$\frac{da_f}{dT} = -i \sum_n a_n(t) \int \varphi_f^* V \varphi_n \, e^{i(E_f - E_n)T} \, d^3x$$

→ before potential act on particle, it is in eigenstate i at $T = -t/2$

$$a_i(-T/2) = 1 \quad , \quad a_n(-T/2) = 0 \quad i \neq n$$

$$\rightarrow \quad \frac{da_f}{dT} = -i \int d^3x \, \varphi_f^* V \varphi_i \, e^{i(E_f - E_i)T}$$

→ assume potential is small and transient and integrate

$$a_f(t) = -i \int_{-t/2}^{t} dT' \int d^3x \, \varphi_f^* V \varphi_i \, e^{i(E_f - E_i)T'}$$

At time $+T/2$ when interactions have ceased:

$$a_f(T/2) \equiv T_{fi} =$$

$$-i \int_{-T/2}^{T/2} dt' \int d^3x \left[\varphi_f(x) e^{-iE_f t} \right]^* V(x,t) \left[\varphi_i(x) e^{-iE_i t} \right]$$

in covariant form:

$$T_{fi} = -i \int d^4x \; \varphi_f^*(x) V(x) \varphi_i(x)$$

good approx. if $a_f(t)$ is small.

Is $|T_{fi}|^2$ the probability that a particle in state i ends up in state f under the influence of V?

Let's take $V(x,t) = V(x)$, we get

$$T_{fi} = \underbrace{-i\int d^3x\, \varphi_f(x)\, V(x)\, \varphi_i(x)}_{V_{fi}} \int_{-\infty}^{\infty} dt\; e^{i(E_f - E_i)t}$$

$$= -2\pi i\, \underbrace{\delta(E_f - E_i)}\, V_{fi}$$

↳ energy conservation during transition

→ uncertainty principle implies that Δt between i and f $= \infty$

→ $|T_{fi}|^2$ not useful ...

We try transition prob. per unit time:

$$W = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$$

We need to square T_{fi}.

remember that $T_{fi} = -i V_{fi} \int_{-\infty}^{\infty} dT \, e^{i(E_f - E_i)T}$

$$= -2\pi i V_{fi} \, \delta(E_f - E_i)$$

$$W = \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{T/2} dT \, e^{i(E_f - E_i)T}$$

applied lim already

$$= \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{T/2} dT$$

$$= 2\pi |V_{fi}|^2 \delta(E_f - E_i)$$

→ need to deal with initial and final states. Here we'll start with well-defined initial state ending up in a set of final states let $\rho(E_f)$ be density of final states

$\rightarrow$ $\rho(E_f)\,dE_f$ is # of final states in energy interval $[E_f, E_f + dE_f]$

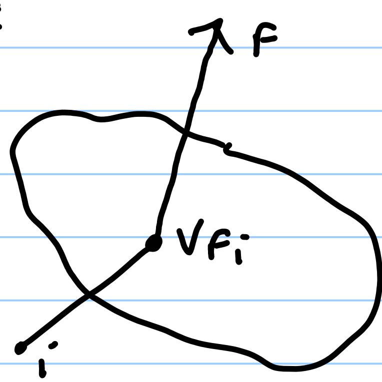
so: $W_{fi} = 2\pi \int dE_f\, \rho(E_f)\, |V_{fi}|^2\, \delta(E_f - E_i)$

$= 2\pi\, |V_{fi}|\, \rho(E_i)$ → Transition prob. per unit Time

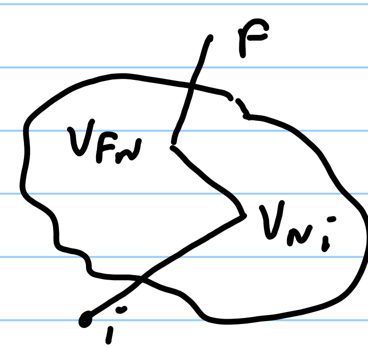
→ Fermi's Golden Rule

Note that the result we obtained was an approx. which we can try to improve.

First order:



second:



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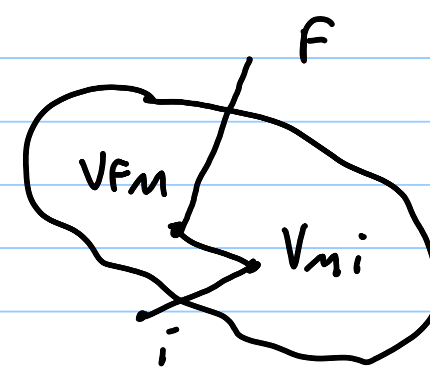

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We go back and insert:

$$a_f(t) = -i\int_{-t/2}^{t} dt' \int d^3x\, \varphi_f^* V \varphi_i\, e^{i(E_f - E_i)t'}$$

which becomes: $-i\int_{-t/2}^{t} dt' \int d^3x\, \varphi_n^* V \varphi_i\, e^{i(E_n - E_i)t'}$

$f \to n$

here

$$\frac{da_f}{dt} = -i\sum_n a_n(t)\int \varphi_f^* V \varphi_n\, e^{i(E_f - E_n)t}\, d^3x$$

and obtain:

$$\frac{da_f}{dt} = \underbrace{-i\int d^3x\, \varphi_f^* V \varphi_i\, e^{i(E_f - E_i)t}}_{\text{First order result above}} \quad \text{or} \quad -iV_{fi}\, e^{i(E_f - E_i)t}$$

$$+ (-i)^2 \left[\sum_{n\neq i} V_{ni}\int_{-T/2}^{T} dt'\, e^{i(E_n - E_i)T'}\right] V_{fn}\, e^{i(E_f - E_n)T}$$

integrate to get T_{fi} $\left(\equiv a_f(t)\right.$ when interactions have cessed$\left.\right)$

first order

(4) $T_{fi} = \ldots + -\sum_{n\neq i} V_{fn}V_{ni} \int_{-\infty}^{\infty} dt\, e^{i(E_f-E_n)t} \int_{-\infty}^{t} dt'\, e^{i(E_n-E_i)t'}$

use $\displaystyle\int_{-\infty}^{t} dt'\, e^{i(E_n-E_i-i\varepsilon)t'} = \frac{i\, e^{i(E_n-E_i-i\varepsilon)t}}{E_i-E_n+i\varepsilon}$

insert into (4)

$$T_{fi} = \ldots -2\pi i \sum_{n\neq i} \underbrace{\frac{V_{fn}V_{ni}}{E_i-E_n+i\varepsilon}}_{\text{propagator}} \delta(E_f-E_i)$$

Term $\perp$ (E_i-E_n)

note that V_{ni} is $\int d^3x\, \varphi_n V \varphi_i$ "vertex factor"

For V, we'll next consider EM interactions

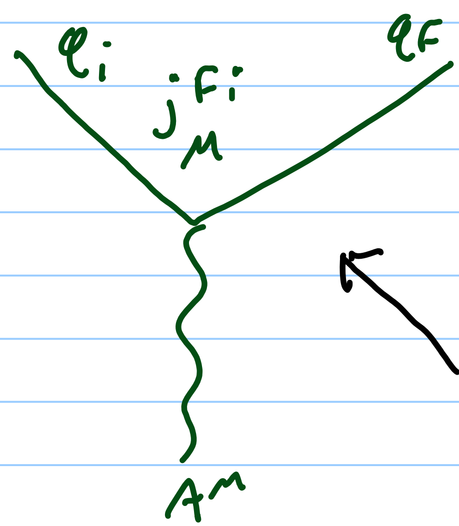
Back To Transition Amplitudes

$$T_{fi} = -i \int \varphi_f^*(x)\, V(x)\, \varphi_i(x)\, d^4x$$

$$= i \int \varphi_f^* \, ie \, (A^\mu \partial_\mu + \underbrace{\partial_\mu}_{\text{acts on } A^\mu \text{ and } \varphi_i} A^\mu)\, \varphi_i \, d^4x$$

int. by parts: $\int \varphi_f^* \partial_\mu (A^\mu \varphi_i) = -\int \partial_\mu (\varphi_f^*) A^\mu \varphi_i$

note $uv \Big|_{-\infty}^{\infty} = 0$ because potential vanishes at ∞



$$T_{fi} = -i \int j_\mu^{fi} A^\mu \, d^4x$$

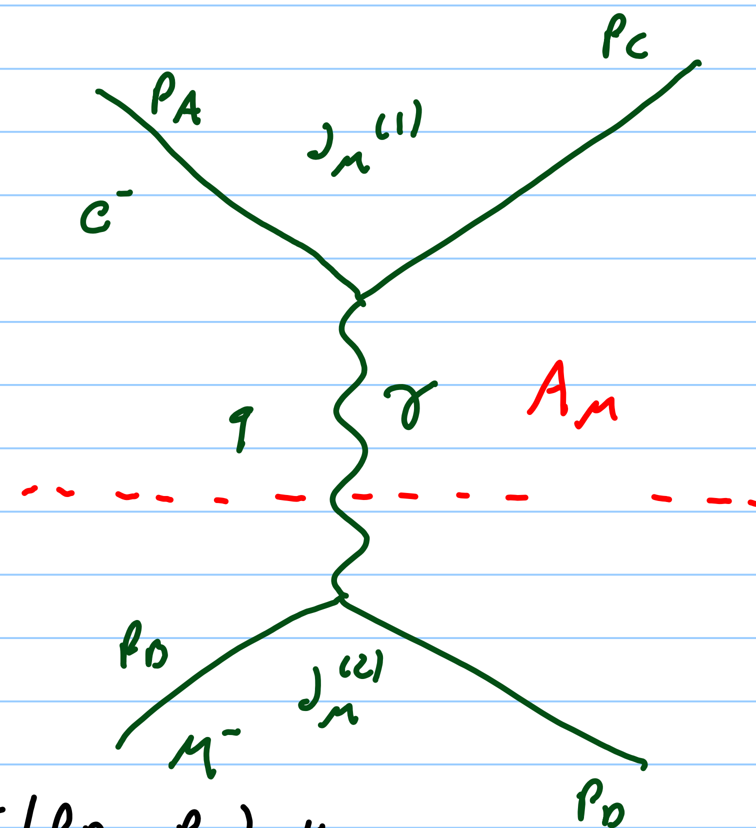
$$j_\mu^{fi}(x) = -ie \left(\varphi_f^* (\partial_\mu \varphi) - (\partial_\mu \varphi_f^*) \varphi_i \right)$$

spinless electron

$$\varphi_i(x) = N_i\, e^{-i p_i \cdot x} \quad , \quad \varphi_f(x) = N_f\, e^{-i p_f \cdot x}$$

$$j_\mu^{fi} = -e N_i N_f (p_i + p_f)_\mu\, e^{i(p_f - p_i)\cdot x}$$

- Electron - Muon Scattering



→ associate A_μ with its source

→ Solve Maxwell equation in Lorentz gauge $(\partial^\mu A_\mu = 0)$

$$\underbrace{\Box^2}_{\partial^\mu \partial_\mu} A^\mu = j^\mu_{(2)}$$

Now: $j^\mu_{(2)} = -e N_B N_D (p_D + p_B)^\mu\, e^{i(p_D - p_B)\cdot x}$

Now: $j^{\mu}_{(2)} = -eN_BN_D(p_D+p_B)^{\mu}\,e^{i(p_D-p_B)\cdot x}$

Because $\Box^2 e^{iq\cdot x} = -q^2 e^{iq\cdot x}$

with $q = (p_D - p_B)$

$\rightarrow \quad A^{\mu} = -\frac{1}{q^2}\, j^{\mu}_{(2)}$

First order this gives:

$$T_{fi} = -i\int j^{(1)}_{\mu}(x)\left(\frac{-1}{q^2}\right) j^{\mu}_{(2)}(x)\, d^4x$$

$T_{fi} = -i\,N_AN_BN_CN_D\,(2\pi)^4\,\delta^{(4)}(p_D+p_C-p_B-p_A)\cdot\mathcal{M}$

with $-i\mathcal{M} = \left(ie\,(p_A+p_C)^{\mu}\right)\left(\frac{-i\,g_{\mu\nu}}{q^2}\right)\left(ie\,(p_B+p_D)^{\nu}\right)$

What do we do with that?

Cross Section Calculation

Step 1 : Fix the normalization

$$\varphi = N e^{-ip\cdot x}$$

prob. density $\rho = 2E|N|^2$

$$\boxed{\rho = i\left(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}\right), \text{ obtained from KG equation}}$$

normalize to $2E$ particles in volume V:

$$\int_V \rho \, dV = 2E \quad \rightarrow \quad N = \frac{1}{\sqrt{V}}$$

Step 2 : obtain transition rate per unit volume

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

with $T_{fi} = -iN_AN_BN_CN_D(2\pi)^4\delta^{(4)}(p_C+p_D-p_A-p_B)\mathcal{M}$

As before, after squaring we get:

$$W_{fi} = (2\pi)^4\delta^{(4)}\frac{(p_C+p_D-p_A-p_B)\,|\mathcal{M}|^2}{V^4}$$

Step 3: obtain cross section from W_{fi}:

$$\text{cross section} = \frac{W_{fi}}{\text{initial flux}} \cdot \text{number of final states}$$

next page

$$= \frac{V\,d^3p}{(2\pi)^3}$$

From particle in 1D box we have

$p_xL = 2\pi N$ or $N = \frac{p_xL}{2\pi}$

$$\frac{\#\text{ of final states}}{\text{particle}} = \frac{V\,d^3p}{(2\pi)^3\cdot 2E}$$

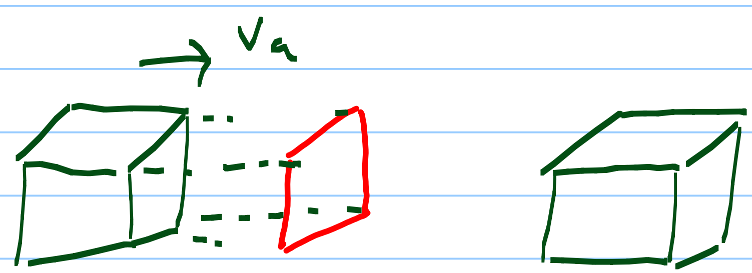
$\Rightarrow$ # of final states : $\dfrac{V\, d^3 p_c}{(2\pi)^3\, 2E_c} \, \dfrac{V\, d^3 p_D}{(2\pi)^3\, 2E_D}$

initial flux:

# of beam particles passing through unit area per unit time :

$$|V_A|\ 2E_A / V$$

# of target particles per unit volume is: $\dfrac{2E_B}{V}$



Initial flux : $|V_A|\, \dfrac{2E_A}{V}\, \dfrac{2E_B}{V}$

Putting everything Together:

$$d\sigma = \frac{V^2}{|v_A|\, 2E_A\, 2E_B} \frac{1}{V^4} |M|^2 \frac{(2\pi)^4}{(2\pi)^6} \delta^{(4)}(p_C + p_D - p_A - p_B) \frac{d^3p_C}{2E_C} \frac{d^3p_D}{2E_D} V^2$$

$\rightarrow V$ goes away $\left[\begin{array}{l}\text{we normalize To } 2E \text{ particles / unit volume} \\ \text{use unit Volume from now on}\end{array}\right]$

we can write $\quad d\sigma = \frac{|M|^2}{F}\, dQ$