LECTURE 1 and 2: Introduction to Gauge Theories

Overview:

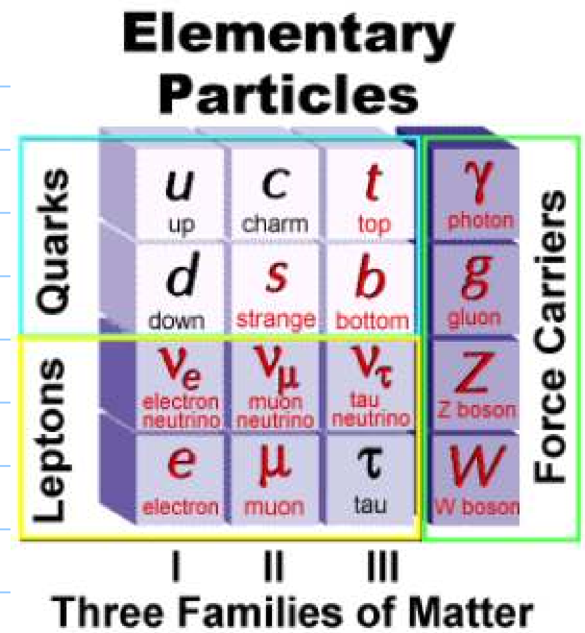
-Overview of Group Theory

-Abelian Gauge Theory

-Non-Abelian Gauge Theory

The Standard Model
A quantum field theory based on SU(3) X SU(2)L X U(1)Y gauge symmetries.

Lagrangian invariant under a continuous group of local transformations (gauge transformations). They form a Lie group whose generators have associated vector fields (gauge fields)



To obtain a better understanding of this theory, we'll spend some time studying what is a gauge theory and this requires some knowledge of group theory. I'll give a brief introduction to group theory and Lie groups in particular. These topics (Group Theory and contruction of Gauge Theories) deserve a course unto themsleves so I encourage you to explore the beautiful mathematics behind our physical theories in more detail during the course of your studies. I suggest H. Georgi's "Lie Algebras in Particle Physics" as a good reference text.

Introduction to Group Theory

"GROUP THEORY IS THE STUDY OF SYMMETRY" Georgi

THE STUDY OF SYMMETRY IN PHYSICAL SYSTEMS IS EXTREMELY USEFUL.

NOETHER'S THEOREM RELATES SYMMETRIES TO CONSERVED QUANTITIES

SPACE TRANSLATION → MOMENTUM

ROTATION → ANG. MOMENTUM

TIME → ENERGY

SYMMETRY → CONSERVED QUANTITIES → INVARIANCE

IN FIELD THEORY, THE ANALOG ARE THE WARD-TAKAHASHI IDENTITIES

UNDERSTANDING HOW SM FIELDS CAN TRANSFORM REQUIRES WE TAKE SOME TIME TO REVIEW SOME GROUP THEORY

Group theory INTRO (cont.)

A group is a set of elements $(a, b, c \dots)$ with a law to combine two elements in an ordered way such that

1- For every $a, b \in G$, the product $ab \in G$
2- $(ab)c = a(bc)$ → associativity
3- G has a unique element e such that for all $a \in G$: $ae = ea = a$ → identity
4- For all $a \in G$, there is a unique element a^{-1} such that $a^{-1}a = aa^{-1} = e$ → inverse

if $ab = ba$: Abelian group → commutative
if $ab \neq ba$: non-Abelian group → non-commutative

examples: following set forms Abelian group $(1, i, -1, -i)$ using multiplication for complex numbers as "combination law"

Note that we can associate matrices to elements above:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

GROUP THEORY

NOTE THAT IN THE CONTEXT OF FIELD THEORY, SYMMETRY TRANSFORMATIONS FORM A GROUP.

EXAMPLE: THEORY WITH ACTION $S[\varphi]$
UNDER A SET OF TRANSFORMATIONS $\{g, h, K, \dots\}$
THE FIELDS CHANGE TO $\varphi_g, \varphi_h, \varphi_K, \dots$
WHICH LEAVES THE ACTION INVARIANT:

$$S[\varphi] = S[\varphi_g] = S[\varphi_h] = \dots$$

ASSUME THAT THE SET $\{g, h, K, \dots\}$ CONTAINS ALL SYMMETRY TRANSFORMATIONS OF THE ACTION.

→ $g \cdot h$ implies succesive application on the fields which will leave action unchanged ⇒ product is in the set

ALSO IDENTITY IS IN THE SET AND SO IS THE INVERSE: $S[\varphi] = S[\varphi_I] = S[\varphi_{g \cdot g^{-1}}] = S[\varphi_{g^{-1}}]$

→ THE FULL SET OF SYMMETRY TRANSFORMATIONS CONSTITUTES A GROUP

GROUP THEO. (cont.)

Def.: A subgroup S, of a group G contains some elements that form a group by themselves.

e and G are trivial subgroups

Def.: The order of a group is the number of elements in G

→ can be infinite

Def.: A representation of G is a mapping D of the elements of G onto a set of linear operators with the following properties:

- $D(e) = 1$, where 1 is the identity operator in the space on which the linear operators act
- $D(g_1)\,D(g_2) = D(g_1 g_2)$ i.e. group combination or multiplication law is mapped onto the multiplication in the space in which the operators act.

Group Theory (cont.)

Example: Z_3

Mult. Table:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

→ abelian

Representation example:

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$[D(g)]_{ij} = \langle e_i | D(g) | e_j \rangle$ → with basis vectors $|e_i\rangle$ associated with e, a, b

Another representation (realization): $D(e) = 1$
$D(a) = e^{2\pi i/3}$
$D(b) = e^{4\pi i/3}$

Group Theory (cont.)

Def: a rep. is said to be reducible if it has an invariant subspace $\rightarrow$ the action of any $D(g)$ on any vector in the subspace is still is the subspace.

In terms of a projection operator P:

$$P\,D(g)\,P = D(g)\,P$$

Example: for Z_3 $\quad P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

we get $D(g)\,P = P$

Def: a completely reducible rep. has its matrix elements in the following form

$$\begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & 0 & \\ \vdots & & \end{pmatrix}$$

i.e. block diagonal

Group Theory (cont.)

Note: a rep. in block diagonal form is said to be in the direct sum of the subrep.

$$D_1 \oplus D_2 \oplus \cdots$$

Def.: A set of generators is a set of group elements such that the repeated application of generators on themselves can generate the whole group

Example: generators: p, q

mult. rules: $p^3 = e$, $q^2 = e$, $(pq)^2 = e$

$\Rightarrow$ Group elements: $\{e, p, p^2, q, qp, qp^2\}$

note that $pqp = q^{-1} = q$

because $q^2 = e$

Lie Groups

We are interested in continuous groups with elements labelled by continuously variable real parameters

$$\alpha_1, \alpha_2, \alpha_3, \dots \alpha_r \quad , \quad g(\alpha_1, \alpha_2, \alpha_3, \dots \alpha_r) \equiv g(\alpha)$$

For a continuous group, condition 1 takes the form:

$$g(\alpha)\, g(\beta) = g(\underbrace{\gamma(\alpha, \beta)}_{\text{continuous functions of } \alpha, \beta})$$

→ if γ is an analytic function, then we have a Lie Group

→ implies that it can be expressed as a power series. We can use a power series expansion to move from one element to another (within its neighborhood).

Lie proved that the properties of the elements which can be reached continuously from I are determined from elements in the neighbourhood of I

Lie Groups (cont.)

see Aitchison and Hey

Consider a group of transformations defined by:

$$x'_i = f_i(x_1, x_2, \dots x_N; \alpha_1, \alpha_2, \dots, \alpha_r)$$

x_i : "coordinates" → or field components on which the transformations act.

α_j : parameters of the transformations

$\alpha = 0$ is the identity transformation $\Rightarrow x_i = f_i(x, 0)$

$$dx_i = \sum_{\nu=1}^{r} \frac{\partial f_i}{\partial \alpha_\nu} d\alpha_\nu$$ (partial deriv. evaluated at $(x,0)$)

$$F \to F + dF = F + \sum_{i=1}^{N} \frac{\partial F}{\partial x_i} dx_i = F + \sum_{i=1}^{N} \left[\sum_{\nu=1}^{r} \frac{\partial f_i}{\partial \alpha_\nu} d\alpha_\nu \right] \frac{\partial F}{\partial x_i}$$

$$\equiv \left[1 - \sum_{\nu=1}^{r} d\alpha_\nu \, i \hat{X}_\nu \right] F$$ with:

$$\boxed{\hat{X}_\nu = i \sum_{i=1}^{N} \frac{\partial f_i}{\partial \alpha_\nu} \frac{\partial}{\partial x_i}}$$ → generator of infinitesimal transfo.

Lie Groups (cont.)

For finite transformations: $\left[1 - \sum_{\nu=1}^{r} d\alpha_\nu \, i\hat{X}_\nu\right]$

becomes $\exp\left[-i\,\alpha\cdot\hat{X}\right]$ $\qquad \alpha\cdot\hat{X} \equiv \sum_\nu \alpha_\nu \hat{X}_\nu$

Theorem states that commutator of any two generators is a linear combination of the generators:

$$[\hat{X}_\alpha, \hat{X}_\beta] = c^{\gamma}_{\alpha\beta}\,\hat{X}_\gamma$$

structure constants of the group
commutation relations called
algebra of the group.

We can see this using the Baker-Campbell-Hausdorff formula. But first we'll take a step back

GROUP THEORY (CONT.)

LET'S TAKE ONE-PARAMETER LIE GROUP WITH ELEMENTS $g(\xi)$. BECAUSE OF ANALYTIC CONDITION, WE CAN CHOOSE A PARAMETRIZATION THAT SATISFIES:

$$g(\xi^1)\, g(\xi^2) = g(\xi^1 + \xi^2)$$

$$\Rightarrow \quad g(0) = I \quad \text{and} \quad g(\xi)^{-1} = g(-\xi)$$

IN THE NEIGHBOURHOOD OF THE IDENTITY WE CAN WRITE

$$g(\xi) = I + \xi T + O(\xi^2)$$

$T \rightarrow$ operator that generates infinitesimal transformations

going finite: $\quad g(\xi) = \{g(\xi/N)\}^N = \lim\limits_{N\to\infty} \{I + \frac{\xi T}{N} \dots\}^N$

$$= \exp(\xi T)$$

For n-parameter $\quad g(\xi_1, \xi_2, \dots) = \exp(\underbrace{\xi_a T^a}_{\text{generators}})$

GROUP THEORY (cont.)

CONSIDER PRODUCT OF TWO ELEMENTS FOR LIE GROUP G:

$$g(\xi_1, \xi_2, \xi_3, \ldots) \cdot g(\zeta_1, \zeta_2, \zeta_3, \ldots) = \exp(\xi_a T^a)\exp(\zeta_b T^b)$$

Baker - Campbell - Hausdorff

$$= \exp\left\{ \xi_a T^a + \zeta_b T^b + \frac{1}{2}\xi_a \zeta_b [T^a, T^b] + \frac{1}{12}(\xi_a \xi_b \zeta_c + \zeta_a \zeta_b \xi_c)[T_a,[T_b,T_c]] + \ldots \right\}$$

SINCE G IS A GROUP WE MUST HAVE:

$$g(\xi_1, \ldots \xi_N) \cdot g(\zeta_1, \ldots, \zeta_N) = \exp(\eta_a T^a)$$

$\rightarrow$ POSSIBLE IF ANY COMMUTATOR CAN BE WRITTEN AS LINEAR COMBINATION OF GENERATORS

$\rightarrow$ GENERATORS MUST CLOSE UNDER COMMUTATION

$$[T^a, T^b] = F_{ab}{}^{c}\, T^c$$

GROUP THEORY (CONT.)

EXAMPLES:

1 – ONE DIM. TRANSLATIONS : $f(x) \to f(x+\xi)$

INF. TRANSFORMATION: $f(x) + \xi \frac{d}{dx} f(x) + O(\xi^2)$

generator : $T f(x) = \frac{d}{dx} f(x)$

FINITE TRANSFO. : $f(x) \to \exp\left(\varepsilon \frac{d}{dx}\right) f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{d^n f(x)}{dx^n}$

↳ Taylor series

2 – TWO DIM. ROTATION : $g(\xi)$ changes θ into $\theta - \xi$

$x = r\cos\theta$, $y = r\sin\theta$: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \xi \begin{pmatrix} y \\ -x \end{pmatrix} + O(\xi^2)$

generator is then $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ note that $T^2 = -I$

Group Theory (cont.)

$$g(\xi) = \exp(\xi t) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n T^n$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n)!} (-\xi^2)^n I + \frac{1}{(2n+1)} (-1)^n \xi^{2n+1} t \right\}$$

$$= \cos\xi \, I + \sin\xi \, t = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}$$

→ Two-dim. rotation

→ The group above is $SO(2)$

"$O(2)$": orthogonal 2×2 matrices

"S" : "special" → $\det = 1$

"$U(N)$": unitary $N\times N$ matrices → N^2 generators

"$SU(N)$": " " " with $\det = 1$

→ have (N^2-1) generators

Let's take a look at $SU(2)$ ...

SU(2)

Consider a system with two fermion fields q_1 and q_2 and demand that it possesses symmetry under transformations that mix them together:

$$q_1 \rightarrow q_1' = \alpha q_1 + \beta q_2$$
$$q_2 \rightarrow q_2' = \gamma q_1 + \delta q_2$$

The $\alpha, \beta, \ldots$ are complex

Keep normalization: $\langle q_i' | q_i' \rangle = \langle q_1 | q_1 \rangle = 1$ etc.

$$\Rightarrow \; |\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$$

Keep orthogonality: $\langle \psi_1' | \psi_2' \rangle = 0$

$$\Rightarrow \; \alpha^* \delta + \beta^* \gamma = 0$$

in 2D component form:

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow q' = \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

SU(2) (cont)

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is unitary : $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A 2×2 unitary matrix has 4 free parameters and we can write:

$$U = e^{i\alpha_0 + \underbrace{i\alpha_1\tau_1 + i\alpha_2\tau_2 + i\alpha_3\tau_3}_{V}}$$

τ_i are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$U = e^{i\alpha_0} V$ with V a member of SU(2)

You can show that V has unit det.

SU(2) (cont.)

For infinitesimal SU(2) Transfo.:

$$\begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \left(1 + i\underbrace{\alpha\cdot\frac{\tau}{2}}\right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\boxed{\frac{i}{2}\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}}$$

$$dq_1 = \frac{i}{2}\alpha_3 q_1 + \left(\frac{i\alpha_1}{2} + \frac{\alpha_2}{2}\right) q_2$$

$$dq_2 = \frac{-i\alpha_3}{2} q_2 + \left(\frac{i\alpha_1}{2} - \frac{\alpha_2}{2}\right) q_1$$

$$\boxed{\hat{X}_\nu = i\sum_{i=1}^{N} \frac{\partial f_i}{\partial \alpha_\nu}\frac{\partial}{\partial x_i}}$$

$$\frac{\partial f_1}{\partial \alpha_1} = \frac{iq_2}{2}, \quad \frac{\partial f_1}{\partial \alpha_2} = \frac{q_2}{2}, \quad \frac{\partial f_1}{\partial \alpha_3} = \frac{iq_1}{2}$$

$$\frac{\partial f_2}{\partial \alpha_1} = \frac{iq_1}{2}, \quad \frac{\partial f_2}{\partial \alpha_2} = -\frac{q_1}{2}, \quad \frac{\partial f_2}{\partial \alpha_3} = -\frac{q_2}{2}$$

$$\hat{X}_1 = \frac{-1}{2}\left\{ q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} \right\}, \quad \hat{X}_2 = \frac{i}{2}\left\{ q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2} \right\}$$

$$\hat{X}_3 = \frac{1}{2}\left\{ -q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2} \right\}$$

$$\boxed{[\hat{X}_i, \hat{X}_j] = i\varepsilon_{ijk}\hat{X}_k}$$

Example 2 : SU(3)

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}' = \left(1 + \frac{i}{2}\eta \cdot \lambda\right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

now we have 8 parameters, and λ are the Gell-Mann matrices e.g. $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

$$\lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

One can determine the generators as in example 2.

SU(3) algebra : $[\hat{G}_a, \hat{G}_b] = i f_{abc} \hat{G}_c$

$f_{123} = 1$, $f_{147} = 1/2$, $f_{458} = \frac{\sqrt{3}}{2}$ etc.

More when we study QCD

Lie Groups (cont.)

We've see particular matrix representations of SU(2) and SU(3). One can use other representations: we just need to respect the group's element multiplication. For SU(2), the triplet representation, which will act on a triplet of fields, must respect the commutation relations. For example, one can write the matrices using $(T^j)_{kl} = -i\varepsilon_{jkl}$

You can check that the 3x3 matrices satisfy the SU(2) commutation relations.

Gauge Invariance and E and M.

Why "gauge"?

Weyl was looking for geometric basis for both E and M and gravity by considering a space-time dependent change of scale.

Eich invarianz: gauge or standard of calibration

consider $F(x)$ that changes between x_μ and $(x_\mu + dx_\mu)$

if space is uniform

$$F(x+dx) = F(x) + \partial^\mu F \, dx_\mu$$

What if the unit of measure or calibration changes in going from x_μ to $(x_\mu + dx_\mu)$?

$$F(x+dx) = (F(x) + \partial^\mu F \, dx_\mu) \cdot (1 + S^\nu dx_\nu)$$

$$F(x+dx) = \left(F(x) + \partial^\mu F\, dx_\mu\right)\cdot\left(1 + S^\nu dx_\nu\right)$$

$$= F(x) + \left(\partial^\mu F(x) + F(x) S^\mu\right) dx_\mu + O(dx)^2$$

$$\Delta F = \underbrace{(\partial^\mu + S^\mu)}_{\text{modified differential operator}} F\, dx_\mu \qquad \text{(First order)}$$

For E and M

$$p_\mu = (E, p_x, p_y, p_z)$$

From QM $\quad p^\mu \to i\partial^\mu \qquad (i\partial^0, -i\nabla)$

$$(p^\mu - eA^\mu) \to i(\partial^\mu + ieA^\mu)$$

if $\quad S^\mu \to ieA^\mu$

$$(1 + ieA^\mu dx_\mu) \simeq e^{ieA^\mu dx_\mu}$$

$\to$ invariance under change of phase (Kept gauge invariance)

Phase invariance in QM

$$\langle O \rangle = \int \psi^* O_{op} \psi$$

$\to$ invariant under overall phase $\psi(x) \to e^{i\alpha}\psi(x)$

but, relative phases do matter

Can we formulate QM with position dependent (local) phase rotations?

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$$

Yes, but at a price

$$\partial_\mu \psi \to \partial_\mu \psi' = e^{i\alpha(x)}\left[\partial_\mu \psi(x) + i(\partial_\mu \alpha(x))\psi(x)\right]$$

picked up extra term ...

replace $\quad \partial_\mu \to D_\mu \equiv \partial_\mu - ieA_\mu$

with $\quad A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$

then

$$D_\mu \psi(x) \to e^{i\alpha(x)} D_\mu \psi(x)$$

quantities like $\psi^* D_\mu \psi(x)$ are invariant under local phase transformations

what did we do? introduced a current

had free field theory $\to$ turned on EM interactions

$i\partial^\mu \to i\partial^\mu + eA^\mu$ at the centre of QED

note that a mass term $m A^\mu A_\mu$ would not preserve local gauge invariance then

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Klein Gordon Equation

we can start with: $E^2 - p^2c^2 = m^2c^4$

$$p^\mu p_\mu - m^2c^2 = 0$$

$p^\mu \to i\hbar\partial^\mu$: $-\hbar^2\partial^\mu\partial_\mu\varphi - m^2c^2\varphi = 0$

$\hbar = c = 1$ from now on...

$$(\partial_\mu\partial^\mu + m^2)\varphi = 0$$

Turn on QED : $i\partial^\mu \to i\partial^\mu + eA^\mu$
$\partial^\mu \to \partial^\mu - ieA^\mu$

$$= (\partial_\mu\partial^\mu + m^2)\varphi = -V\varphi$$

$$V = -ie(\partial_\mu A^\mu + A^\mu\partial_\mu) - e^2A^2$$

$\alpha = \frac{e^2}{4\pi} \simeq 1/137$ $\to$ small $\to$ pert. theory

lowest order, omit e^2 Term

Non-Abelian gauge theories

For the Abelian case we studied, $U(1)$, we saw that imposing local gauge transformations:

$\varphi(x) \to e^{i\alpha(x)}\varphi(x)$ required addition of gauge-covariant derivative to keep theory invariant under those transformations: $D_\mu \equiv \partial_\mu + iq A_\mu(x)$

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha$$

We now look at an example of a non-Abelian gauge theory: $SU(2)$-isospin gauge theory

$$\psi = \begin{pmatrix} \text{proton} \\ \text{neutron} \end{pmatrix}$$

Global symmetry $\to$ freedom to choose what we call a proton and neutron everywhere

Non-Abelian gauge theories

Consider a local gauge transformation for the field $\psi(x)$:

$$\psi(x) \rightarrow \psi(x)' = G(x)\psi(x)$$

with $$G(x) \equiv \exp\left(\frac{i}{2}\tau \cdot \alpha(x)\right)$$

τ are the Pauli matrices

$\alpha \equiv \alpha_1, \alpha_2, \alpha_3$

$$\partial_\mu \psi \rightarrow G(\partial_\mu \psi) + \underline{(\partial_\mu G)\psi}$$

To take care of this term, let's introduce a gauge covariant derivative:

$$D_\mu \equiv I\partial_\mu + igB_\mu$$

with $\underline{B_\mu} = \frac{1}{2}\tau^a b_\mu^a = \frac{1}{2}\begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

2x2 matrix

$b_\mu = (b_1, b_2, b_3)$

Non-Abelian gauge theories (cont.)

How does B_μ need to transform to cancel extra terms?

We want $D_\mu \psi \to D_\mu' \psi' = \underbrace{G(D_\mu \psi)}$

$$D_\mu' \psi' = (\partial_\mu + ig B_\mu')\psi'$$

$$= G(\partial_\mu \psi) + (\partial_\mu G)\psi + ig B_\mu'(G\psi)$$

$$\equiv G(\partial_\mu + ig B_\mu)\psi$$

$$= G(\partial_\mu \psi) + ig\, G(B_\mu \psi)$$

$$\Rightarrow \quad ig B_\mu'(G\psi) = ig\, G(B_\mu \psi) - (\partial_\mu G)\psi$$

multiply from the right by G^{-1}

$$\to \quad B_\mu' = G B_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}$$

$$= G\left[B_\mu + ig\, G^{-1}(\partial_\mu G)\right]G^{-1}$$

Non-Abelian gauge theories (cont.)

$$B_\mu' = G\left[B_\mu + \frac{i}{g} G^{-1}(\partial_\mu G)\right] G^{-1} \qquad (1)$$

Looks complicated... let's try it with $G_{em} = e^{iq\alpha(x)}$

We get: $A_\mu' = e^{iq\alpha(x)}\left[A_\mu + \frac{i}{q} \cdot e^{-iq\alpha(x)} \cdot iq\,\partial_\mu \alpha\, e^{iq\alpha(x)}\right] e^{-iq\alpha(x)}$

$= A_\mu - \partial_\mu \alpha$ (ok it works...)

Consider infinitesimal gauge transformation ($|\alpha_i| \ll 1$)

$G = 1 + \frac{i}{2}\tau\cdot\alpha$ using (1) above we get:

$G B_\mu G^{-1} = \left[B_\mu + \frac{i}{2}\tau\cdot\alpha B_\mu\right] \cdot 1 - \frac{i}{2}\tau\cdot\alpha$

$= B_\mu + \frac{i}{2}\tau\cdot\alpha\, B_\mu - \frac{i}{2} B_\mu\, \tau\cdot\alpha + O(\alpha^2)$

$\frac{i}{g}(\partial_\mu G) G^{-1} = \left[\frac{i}{g}\,\frac{i}{2}\,\partial_\mu(\alpha\cdot\tau)\right] \cdot \left(1 - \frac{i}{2}\tau\cdot\alpha\right) = -\frac{1}{2g}\,\partial_\mu(\alpha\cdot\tau)$

Non-Abelian gauge theories (cont.)

we have

$$B'_\mu = B_\mu + \frac{i}{2}\alpha\cdot\tau B_\mu - \frac{i}{2}B_\mu \alpha\cdot\tau - \frac{1}{2g}\partial_\mu(\alpha\cdot\tau) + \ldots$$

now $B_\mu = \frac{1}{2}\tau\cdot b_\mu$ so we get:

$$\tau\cdot b'_\mu = \tau\cdot b_\mu + \frac{i}{2}\underbrace{(\tau\cdot\alpha\,\tau\cdot b_\mu - \tau\cdot b_\mu\,\alpha\cdot\tau)} - \frac{1}{g}\partial_\mu(\alpha\cdot\tau)$$

in component form: $\frac{i}{2}\alpha^j b^k_\mu(\tau^j\tau^k - \tau^k\tau^j) = \frac{i}{2}\alpha^j b^k_\mu[\tau^j,\tau^k]$

For $SU(2)$: $[\tau^j,\tau^k] = 2i\varepsilon_{jkl}\tau^l$

second term becomes: $-\varepsilon_{jkl}\alpha^j b^k_\mu \tau^l = -\alpha\times b_\mu\cdot\tau$

isospin components are linearly indep. so:

$\tau\cdot b'_\mu = \tau\cdot b_\mu - \alpha\times b_\mu\cdot\tau - \frac{1}{g}\partial_\mu(\alpha\cdot\tau)$ becomes:

$$b'^l_\mu = b^l_\mu - \underbrace{\varepsilon_{jkl}\alpha^j b^k_\mu} - \frac{1}{g}\partial_\mu\alpha^l$$

Non-Abelian gauge theories (cont.)

Note: new term $[\varepsilon_{jkl}\, \alpha^j\, b^k_\mu]$ is picked because gauge transformations do not commute

Now we focus on field strength tensor. We would like it to transform as:

$$F'_{\mu\nu} = G F_{\mu\nu} G^{-1}$$

$F_{\mu\nu} \rightarrow F^a_{\mu\nu}\, \tau^a$

$\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$

and $\mathrm{Tr}(\tau^a \tau^b) = 2\delta^{ab}$

if we try

$\partial_\nu B'_\mu - \partial_\mu B'_\nu$ for $F'_{\mu\nu}$

we do not get

$$G(\partial_\nu B_\mu - \partial_\mu B_\nu) G^{-1}$$

Are we missing a commutator again?

Non-Abelian gauge theories (cont.)

if we try $F_{\mu\nu} = \frac{1}{iq}[D_\nu, D_\mu]$ for QED:

$$F_{\mu\nu} = \frac{1}{iq}[(\partial_\nu + iqA_\nu), (\partial_\mu + iqA_\mu)]$$

$$= \partial_\nu A_\mu - \partial_\mu A_\nu + iq\underbrace{[A_\nu, A_\mu]}_{=0}$$

$$F_{\mu\nu} = \frac{1}{ig}[D_\nu, D_\mu] = \partial_\nu B_\mu - \partial_\mu B_\nu + ig[B_\nu, B_\mu]$$

$$G F_{\mu\nu} G^{-1} = F'_{\mu\nu}$$

Yang-Mills Lagrangian: $\bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$

in component form: $F^\ell_{\mu\nu} = \partial_\nu B^\ell_\mu - \partial_\mu B^\ell_\nu + g\,\varepsilon_{jk\ell}\, b^j_\mu b^k_\nu$

$$b'^\ell_\mu = b^\ell_\mu - \frac{1}{g}\partial_\mu \alpha^\ell - \varepsilon_{jk\ell}\,\alpha^j b^k_\mu$$

For other groups, we'll replace $\varepsilon_{jk\ell}$ by the group's structure constants $f_{jk\ell}$

Non-Abelian gauge theories (cont.)

$$g_1 = 1 + i w_1^\alpha T_\alpha \qquad g_2 = 1 + i w_2^\alpha T_\alpha\,, \quad g_1 g_2 = 1 + i(w_1^\alpha + w_2^\alpha) T_\alpha + \ldots$$

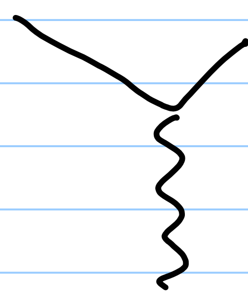
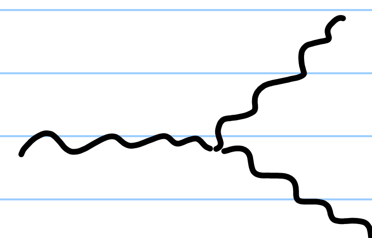
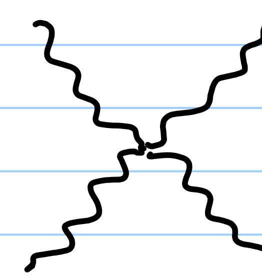
$$g_1 g_2 (g_2 g_1) \approx (1 + i w_1^\alpha T_\alpha)(1 + i w_2^\beta T_\beta)(1 - i w_2^\beta T_\beta)(1 - i w_1^\alpha T_\alpha)$$
$$= 1 - w_1^\alpha w_2^\beta [T_\alpha, T_\beta] \qquad , \qquad [T_\alpha, T_\beta] = i f_{\alpha\beta}^{\gamma} T_\gamma$$

see Appendix B of Burgess and Moore

Also note that the non-Abelian theory has additional couplings:

$$F \propto (\partial B - \partial B) + g BB$$

$$\mathcal{L} \propto \underbrace{(\partial B - \partial B)}_{\text{propagator}} + g \underbrace{(\partial B - \partial B) BB}_{\text{triple}} + \underbrace{g^2 BBBB}_{\text{quartic}}$$

U(1) and SU(2) recap

U(1): $\varphi \to \varphi' = \exp(i\alpha(x))\,\varphi$

SU(2) $\varphi \to \varphi' = \exp\left(\frac{i}{2}\tau \cdot \alpha(x)\right)\varphi$

$\tau \hookrightarrow$ 3 Pauli Matrices; $\alpha(x) \hookrightarrow \alpha_1, \alpha_2, \alpha_3$

Both cases: $\partial_\mu \varphi \to G(\partial_\mu \varphi) + \underbrace{(\partial_\mu G)\varphi}$

To Take care of extra Term introduce gauge covariant derivative:

U(1): $D_\mu = \partial_\mu + ieA_\mu$

SU(2): $D_\mu = I\partial_\mu + igB_\mu$ $\quad B_\mu = \frac{1}{2}\tau \cdot b_\mu$, $b_\mu \hookrightarrow b_\mu^1, b_\mu^2, b_\mu^3$

Both cases: $C_\mu' = G C_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}$

U(1): $A_\mu \to A_\mu' = A_\mu - \frac{1}{e}\partial_\mu \alpha$

SU(2): $b_\mu \to b_\mu' = b_\mu - \alpha \times b_\mu - \frac{1}{g}\partial_\mu \alpha$

or $b_\mu^{\ell\prime} = b_\mu^\ell - \varepsilon_{jk\ell}\alpha^j b^k - \frac{1}{g}\partial_\mu \alpha^\ell$