Appendix

C

Introduction to group theory

Invariance groups

A group G is a set of elements $\{g, h, k, ...\}$ for which a multiplication is defined which assigns to *every* two elements g, hG an element $g \cdot h$ which is again an element of the group. In addition the following properties should hold:

(i) The multiplication is *associative*, which means that we have $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for all g, h, kG. In the special case that the group multiplication is commutative, $g \cdot h = h \cdot g$ for all g, hG, the group is called *abelian.*

(2) The set of elements of G contains the identity $\mathbf{I}$, for which we have $\mathbf{I}{\cdot}g = g{\cdot}\mathbf{I}$ for all gG, as well as the inverse elements g^{-1} for every gG, i.e. $g^{-1} \cdot g = g \cdot g^{-1} = \mathbf{I}$. A subset H of elements contained in G is called a *subgroup* of G if H itself is also a group according to the definition given above.

Symmetry transformations always form a group. To see this consider for instance a theory described in terms of an action $S[\phi]$ which is a functional of fields generically denoted by ϕ. Under a set of symmetry transformations $\{g, h, k, \ldots\}$ the fields ϕ change into $\phi_g, \phi_h, \phi_k, \ldots$ which leave the action invariant, i.e.

$$S[\phi_g] = S[\phi_h] = S[\phi_k] = \cdots = S[\phi]. \tag{C.1}$$

We assume that the set $\{g, h, k, \ldots\}$ is complete in the sense that it contains all symmetry transformations of the action. The effect of a product of two such transformations, say $g \cdot h$, follows from successive application of g and h to the fields, first h and then g. Clearly the action remains invariant under the product transformation as well, which must therefore be one of the elements of the set $\{g, h, k, ...\}$. Furthermore the identity transformation is contained in this set, from which it follows that also the inverse transformations define symmetries of the action, since

$$S[\phi] = S[\phi_{\boldsymbol{I}}] = S[\phi_{g \cdot g^{-1}}] = S[\phi_{g^{-1}}]\,, \tag{C.2}$$

where the last step follows from (C.l) after replacing ϕ by $\phi_{g^{-1}}$, (we assume here that all transformations $\phi \to \phi_g$ have an inverse). Consequently the defining properties of a group are satisfied, so that the full set of symmetry transformations constitutes a group.

One can make an obvious distinction between discrete and continuous transformations. *Discrete* symmetries usually constitute a finite group, i.e. a group consisting of a finite number of elements. An example of a discrete symmetry is a reflection; as the product of two reflections gives the identity the corresponding group consists of precisely two elements, namely the identity transformation and the reflection. Another example is the group of rotations that leave a three-dimensional cube invariant. This group has 24 elements. *Continuous* symmetries depend on one or more parameters in a continuous fashion. Clearly a group of such transformations contains an infinite number of elements. The dimension of a continuous group is defined as the number of independent parameters on which the group elements depend. If the dependence on these parameters is analytic then we are dealing with a so-called *Lie group*. The three-dimensional rotations constitute a well-known example of a Lie group; they depend on the three Euler angles. Consequently the dimension of the rotation group is 3.

If there is a mapping from a group G to a set of matrices $D(G)$ which preserves the group multiplication then $D(G)$ is called a *representation* of the group G. In that case, to any element gG there belongs a matrix $D(g)D(G)$ such that to the product $g \cdot h$ of two elements g and h of G there belongs a matrix $D(g \cdot h)$ such that

$$D(g \cdot h) = D(g)D(h)D(G)\,. \tag{C.3}$$

The mapping between G and $D(G)$ is called a homomorphism. If the mapping is one-to-one then it is called an isomorphism and $D(G)$ is a faithful representation. In case the mapping is not into a set of matrices but into some other algebraic structure, it is called a realization. In general a group can have many different representations.

As an example we recall the representations of the rotation group, which are well-known from quantum mechanics. These representations are characterized by an integer $l(l = 0,1,2,...)$ and consist of $(2l+1) \times (2l+1)$ matrices acting on states with total angular momentum $L^2 = \hbar^2 l(l+1)$; the latter are labelled by their value of angular momentum projected along a certain axis (e.g. $L_z = -\hbar l, -\hbar(l-1), .., \hbar l$). For each rotation g (which is a 3×3 orthogonal matrix) there is a $(2l+1) \times (2l+1)$ matrix $D(g)$, which specifies how the $2l+1$ states transform among themselves as a result of the rotation. The quantity $2l+1$ is called the dimension of the representation.

It is rather obvious that combining two representations of dimension $2l_1+1$ and $2l_2+1$ leads to another representation of dimension $2(l_1+l_2+1)$. The latter representations are called *reducible* as they can be reduced to smaller representations. Evidently not much new is to be learnt from studying reducible representations, so that one usually restricts oneself to *irreducible* representations. It can be shown that finite groups must have a finite number of irreducible representations (we will exploit this fact in appendix E). Continuous groups have infinitely many representations, which can fortunately be studied rather systematically as one sees from the example of the rotation group. Lie groups

Consider a one-parameter Lie group with elements $g(\xi)$. Because of the analyticity of $g(\xi)$ it is always possible to choose a so-called canonical parametrization, which satisfies

$$g(\xi^1)g(\xi^2) = g(\xi^1 + \xi^2)\,. \tag{C.4}$$

Consequently

$$g(0) = \mathrm{I}, \quad (g(\xi)^{-1} = g(-\xi)\,. \tag{C.5}$$

Using this parametrization and the fact that $g(\xi)$ is analytic we can write an element in a neighbourhood of the identity element as

$$g(\xi) = \mathrm{I} + \xi t + O(\xi^2)\,, \tag{C.6}$$

where t is an operator which generates the infinitesimal group transformation. Using (C.4) we can formally construct finite elements $g(\xi)$ by making an infinite series of infinitesimally small steps away from the identity element:

$$g(\xi) = \{g(\xi/n)\}^n = \lim_{n\to\infty}\left\{\mathrm{I} + \frac{\xi}{n}t\ldots\right\}^n = \exp(\xi t)\,, \tag{C.7}$$

where the exponentiation is defined by its series expansion. The result (C.7) can directly be extended to an n-parameter Lie group:

$$g(\xi^1,\ldots,\xi^n) = \exp(\xi^a t_a)\,, \tag{C.8}$$

where we have adopted the summation convention. The quantities t_a, which characterize the infinitesimal transformations that are linearly independent, are called the *generators* of the Lie group. For compact groups one can show that every group element can be written in the form (C.8) (for compact groups the (group-invariant) "volume" of the parameter space is finite).

To elucidate these notions let us discuss two examples. First consider the set of all one-dimensional translations: $g(\xi)$ is then the transformation that changes the coordinate x into $x+\xi$. Obviously, these transformations form a group with a canonical parameter ξ. In the space of functions of one variable, a representation of this group is given by the transformation that changes any function $f(x)$ according to:

$$f(x) \to f(x+\xi)\,. \tag{C.10}$$

An infinitesimal transformation is then given by

$$f(x) \to f(x+\xi) = f(x) + \xi\frac{\mathrm{d}}{\mathrm{d}x}f(x) + O(\xi^2)\,, \tag{C.10}$$

so that the generator of the translation group is

$$tf(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) \,. \tag{C.11}$$

Indeed,a finite transformation can be written as

$$f(x) \to \exp\left(\xi \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} \,. \tag{C.12}$$

which is simply $f(x+\xi)$ expanded as a Taylor series about x.

As a second example consider all two-dimensional rotations, which obviously form a Lie group with the angle of rotation ξ as a natural canonical parameter. Using polar coordinates with $x = r\cos\theta, y = r\sin\theta$ a (clockwise) rotation $g(\xi)$ changes the value θ into $\theta - \xi$. Infinitesimally one has

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} + \xi \begin{pmatrix} y \\ -x \end{pmatrix} + O(\xi^2) \,. \tag{C.13}$$

According to $(C.13)$ the generator t can be written as a 2×2 matrix

$$t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,. \tag{C.13}$$

Using $t^2 = -\boldsymbol{I}$ a finite rotation $g(\xi)$ can be written as

$$\begin{aligned} g(\xi) &= \exp(\xi t) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n t^n \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n)!} (-\xi^2)^n \mathrm{I} + \frac{1}{(2n+1)!} (-1)^n \xi^{2n+1} t \right\} \\ &= \cos\xi \mathrm{I} + \sin\xi t = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \end{aligned} \tag{C.15}$$

which indeed constitutes a general two-dimensional rotation. The above group is called SO(2). It is the group of all orthogonal 2×2 matrices with unit determinant. This is a special case of the group O(N) which consists of all orthogonal $N \times N$ matrices, and the group SO(N) for the subgroup of elements of O(N) with unit determinant. Similarly, U(N) is the group of unitary $N \times N$ matrices, and SU(N) is the group of elements of U(N) with unit determinant. Obviously, O(N) and SO(N) are subgroups of U(N) and SU(N), respectively. Lie algebra

Let us now return to a more general discussion of the group generators. Consider an n-parameter Lie group G with elements $g(\xi^1, \ldots, \xi^n)$ and generators $t_a (a = 1, \ldots, n)$. According to (C.8) we may write

$$g(\xi^1, \ldots, \xi^n) = \exp(\xi^a t_a) \, .$$

A product of two such elements can be expressed by means of the Baker-Campbell-Hausdorff formula

$$\begin{aligned} & g(\xi^1, \ldots, \xi^n) \cdot g(\zeta^1, \ldots, \zeta^n) = \exp(\xi^a t_a) \cdot \exp(\zeta^b t_b) \\ & = \exp\{\xi^a t_a + \zeta^a t_a + \tfrac{1}{2}\xi^a \zeta^b [t_a, t_b] + \tfrac{1}{12}(\xi^a \xi^b \zeta^c + \zeta^a \zeta^b \xi^c [t_a, [t_b, t_c]] \\ & + \text{higher} - \text{order} \quad \text{commutators} \quad \text{of} \quad \text{the} \quad t'\text{s}\} \, . \end{aligned} \qquad (C.16)$$

Because G is a group, the product (C.16) must again be an exponential form of the generators, so there must be coefficients $\eta^1, \ldots, \eta^n$ such that

$$g(\xi^1, \ldots, \xi^n) \cdot g(\zeta^1, \ldots, \zeta^n) = \exp(\eta^a t_a) \, . \qquad \text{(C.17)}$$

This is possible if and only if any commutator of generators can again be written as a linear combination of generators. In other words, the generators must *close under commutation*:

$$[t_a, t_b] = f_{ab}{}^c t_c \, , \qquad \text{(C.18)}$$

where $f_{ab}{}^c$ are constants, antisymmetric in their lower indices (since we assume real parameters ξ^a, these constants are real). With this property the generators t_a form the basis of the so-called *Lie algebra g* associated with the Lie group G. From (C.16) it follows that the $f_{ab}{}^c$ determine the multiplication table of the Lie group. Therefore they are called the *structure constants* of the group (observe that abelian groups have zero structure constants). Two groups with the same structure constants are *locally* equivalent, in the sense that there is a one-to-one correspondence between the elements of the two groups in a neighbourhood of any group element. This does not necessarily imply that there is a one-to-one correspondence between the two groups as a whole, i.e. the groups need not be *globally* equivalent. To understand the difference between the two equivalency relations, consider a circle and a line. In a small neighbourhood of any point, the circle and the line admit a one-to-one mapping of the points of one into those of the other. However, globally there exists no such mapping because the (unit) circle is equivalent to a line of which all elements modulo distances of 2π are considered as identical. Consequently the circle is not simply connected because not all closed paths can be deformed continuously to a point. As we shall see later a similar phenomenon happens for groups.

As an example consider the group SU(2), defined as the set of all unitary 2×2 matrices with unit determinant. Elements of this group can be written as

$$g_{\mathrm{SU}(2)}(\boldsymbol{\xi}) = \exp(\xi^a t_a)\,. \tag{C.19}$$

The requirement that $g_{\mathrm{SU}(2)}(\boldsymbol{\xi})$ is a unitary matrix with unit determinant leads to the following two conditions for the generators:

$$t_a = -t_a^\dagger\,, \mathrm{Tr}(t_a) = 0\,. \tag{C.20}$$

Therefore we can write the SU(2) generators in terms of the three Pauli matrices:

$$t_a = \tfrac{1}{2}\mathrm{i}\tau_a\,, \qquad \text{with}$$
$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.21}$$

The structure constants follow directly from

$$[t_a, t_b] = -\tfrac{1}{4}[\tau_a, \tau_b] = -\tfrac{1}{4}(2\mathrm{i}\varepsilon_{abc}\tau_c) = -\varepsilon_{abc}t_c\,. \tag{C.22}$$

Finite elements of SU(2) can again be found by exponentiation. From the well-known identity

$$\tau_a\tau_b + \tau_b\tau_a = 2\delta_{ab}\mathrm{I}\,, \tag{C.23}$$

it follows that $(\xi^a\tau_a)^2 = \xi^2\mathrm{I}$, where we use the definition

$$\xi = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}\,, \tag{C.24}$$

so that

$$\begin{aligned} g_{\mathrm{SU}(2)}(\boldsymbol{\xi}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\xi^a t^a)^n \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n)!}(\tfrac{1}{2}\mathrm{i}\xi)^{2n}\mathrm{I} + \frac{1}{(2n+1)!}(\tfrac{1}{2}\mathrm{i}\xi)^{2n+1}\Big(\xi^a/\xi\Big)\tau_a \right\} \\ &= \cos\tfrac{1}{2}\xi\mathrm{I} + \mathrm{i}\frac{\sin\frac{1}{2}\xi}{\xi}\xi^a t_a \\ &= \begin{pmatrix} \cos\frac{1}{2}\xi + \mathrm{i}\frac{\sin\frac{1}{2}\xi}{\xi}\xi^3 & \mathrm{i}\frac{\sin\frac{1}{2}\xi}{\xi}(\xi^1 - \mathrm{i}\xi^2) \\ \mathrm{i}\frac{\sin\frac{1}{2}\xi}{\xi}(\xi^1 + \mathrm{i}\xi^2) & \cos\frac{1}{2}\xi - \mathrm{i}\frac{\sin\frac{1}{2}\xi}{\xi}\xi^3 \end{pmatrix} \end{aligned} \tag{C.25}$$

Fig. C.1. Slice of the parameter space of the group SU(2) with $\xi^3 = 0$. The origin corresponds to the identity $\mathbf{I}$. The boundary is a circle with radius 2π, which corresponds to the SU(2) element $g = -\mathbf{I}$. Each element inside the inner circle, which has radius π has a corresponding element in the outer region which differs by an overall minus sign according to (C.27). Three pairs of such points are indicated. The solid curve connecting two opposite points on the inner circle corresponds to a continuous set of SU(2) transformations with endpoints corresponding to two transformations differing by an overall sign. The parameter space of SO(3) can be imbedded in the same plot and covers only the inside of the circle with radius π. Two opposite points on the inner circle correspond to identical SO(3) elements. The SO(3) elements corresponding to the solid curve therefore describe a *closed* continuous set of SO(3) transformations.

Because of the periodic dependence of (C.25) on ξ the parameter space of SU(2) can be restricted to the inside of a 2-dimensional sphere of radius 2π, i.e.

$$(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \leqslant (2\pi)^2 \,. \tag{C.26}$$

The parameter space of SU(2), a slice of which is shown in fig. C.l, can be divided into two parts: an inside region where $\xi \leqslant \pi$ and an outer shell with $\pi < \xi \leqslant 2\pi$. To each point ξ in the first region one can assign a point ξ' in the second region, such that both are located on a straight line passing through the origin and separated by a distance 2π (so that they are in opposite directions); explicitly

$$\boldsymbol{\xi}' = -\frac{2\pi - \xi}{\xi}\boldsymbol{\xi}\,, 0 \leqslant \xi \leqslant \pi\,, \pi \leqslant \xi' \leqslant 2\pi\,. \tag{C.27}$$

The SU(2) elements corresponding to $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are then related by

$$g_{\mathrm{SU}(2)}(\boldsymbol{\xi}') = -g_{\mathrm{SU}(2)}(\boldsymbol{\xi})\,. \tag{C.28}$$

All points on the boundary of parameter space (a 2-dimensional sphere with radius 2π correspond to the *same* group element $g_{\mathrm{SU}(2)} = -\boldsymbol{I}$ [the boundary and the origin are thus related according to (C.28)]. Therefore the SU(2) group manifold has the topology of a 3-dimensional sphere (i.e. a sphere in 4-dimensional Euclidean space), in the same way as identifying the points on the boundary of a 2-dimensional disc leads to a 2-dimensional sphere in 3-dimensional Euclidean space (see problem C.1).

To appreciate the differences let us also consider a comparable non-compact group. Such a group is Sl(2, R), the group of 2×2 real matrices with unit determinant. Using the relation

$$g_{\mathrm{Sl}(2,\mathrm{R})}(\xi) = \exp(\xi^a t_a)\,, \tag{C.29}$$

one finds that the three generators of this group are

$$t_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad t_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \quad t_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{C.30}$$

The structure constants follow from the nonvanishing commutators

$$[t_1, t_2] = t_3\,, \quad [t_2, t_3] = t_1\,, \quad [t_3, t_1] = -t_2\,, \tag{C.31}$$

which almost coincide with the commutators (C.22) of the SU(2) generators. Finite elements can be constructed by using (C.29). The analogue of (C.23) takes the form

$$t_a t_b + t_b t_a = \tfrac{1}{2}\eta_{ab}\mathrm{I}\,, \tag{C.32}$$

with η_{ab} a 3×3 diagonal matrix with eigenvalues $(+1, -1, +1)$. From this relation it follows that $(\xi^a t_a)^2 = \frac{1}{4}\xi^2\mathrm{I}$, where we now use the definition

$$\xi = \sqrt{(\xi^1)^2 - (\xi^2)^2 + (\xi^3)^2}\,. \tag{C.33}$$

The exponentiation of $\xi^a t_a$ leads to

$$\begin{aligned} g_{\mathrm{Sl}(2,\mathrm{R})}(\boldsymbol{\xi}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\xi^a t_a)^n \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n)!} (\tfrac{1}{2}\xi)^{2n} \mathrm{I} + \frac{1}{(2n+1)!} (\tfrac{1}{2}\xi)^{2n+1} \Big(2\xi^a/\xi \Big) t_a \right\} \\ &= \cosh \tfrac{1}{2}\xi \mathrm{I} + \frac{\sinh \frac{1}{2}\xi}{\xi} \xi^a t_a \\ &= \begin{pmatrix} \cosh \frac{1}{2}\xi + \frac{\sinh \frac{1}{2}\xi}{\xi}\xi^3 & \frac{\sinh \frac{1}{2}\xi}{\xi}(\xi^1 - \xi^2) \\ \frac{\sinh \frac{1}{2}\xi}{\xi}(\xi^1 + \xi^2) & \cosh \frac{1}{2}\xi - \frac{\sinh \frac{1}{2}\xi}{\xi}\xi^3 \end{pmatrix} \end{aligned} \qquad (C.25)$$

There are obvious similarities between the results for SU(2) and Sl(2, R), but there are also important differences. One difference is that the volume of the parameter space of Sl(2, R) is infinite, which is a typical feature of non-compact groups. If $\xi^2 = \xi^a \eta_{ab} \xi^b$ is negative we must replace ξ by $\mathrm{i}|\xi|$, so that (C.32) becomes periodic in ξ. Therefore the parameters can be restricted to

$$\xi^a \eta_{ab} \xi^b = (\xi^1)^2 - (\xi^2)^2 + (\xi^3)^2 \geqslant -(2\pi)^2 \,. \qquad (C.35)$$

The parameter space of Sl(2, R) is shown in fig. C.2, where we have indicated the elements of the SO(2) subgroup that Sl(2, R) has in common with SU(2). The boundary of the parameter space formed by the hyperboloid corresponds to the same group element $g_{\mathrm{Sl}(2,\mathrm{R})} = -\mathbf{I}$.

A second aspect of noncompact groups is that exponentiation of the generators does not lead to all possible group elements. For instance, the matrix

$$g = \begin{pmatrix} -e^{\alpha} & 0 \\ 0 & -e^{-\alpha} \end{pmatrix} \qquad (C.36)$$

is an element of Sl(2, R), although it cannot be written in the form (C.34). However, all elements of Sl(2, R) can be obtained as a product of a finite number of elements which can each be exponentiated. For instance, (C.36) can be written as the product of two elements of type (C.34), i.e.

$$g = g(\xi^1 = 0, \xi^2 = 2\pi, \xi^3 = 0) \cdot g(\xi^1 = 0, \xi^2 = 0, \xi^3 = 2\alpha) \,. \qquad (C.37)$$

Representations

Suppose that one can find n matrices Y_a with the same commutation relations as the elements of some Lie algebra g:

$$[Y_a, Y_b] = f_{ab}{}^c Y_c \,, \qquad (C.38)$$

Fig. C.2. The parameter space of the group Sl(2, R). The origin corresponds to the identity **I**. The boundary is a hyperboloid which corresponds to the Sl(2,R) element $g = -\mathbf{I}$. Elements corresponding to the line with ξ^2 between -2π and 2π and $\xi^1, \xi^3 = 0$ constitute the (compact) subgroup SO(2).

then, by definition, these matrices form the basis of a *representation* of this Lie algebra. Exponentiation of linear combinations of Y_a leads to a representation of the corresponding Lie group G:

$$g(\xi^1, \ldots, \xi^n) \to \exp(\xi^a Y_a)\,, \tag{C.39}$$

because the ${f_{ab}}^c$ completely determine the multiplication table of the Lie group. Thus, each representation of the Lie algebra induces a representation of the corresponding group. Furthermore the quantities $-Y_a^\dagger$ also satisfy (C.38), thus defining a second representation which may or may not be equivalent to the first one.

For each Lie group there is a special representation called the *adjoint representation*, which has the same dimension as the group itself. This follows from the *Jacobi identity* which holds for any three matrices A, B and C:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \tag{C.40}$$

Choosing $A = t_a, B = t_b, C = t_c$ and using (C.18) one obtains the Jacobi identity for the structure constants

$${f_{ab}}^e {f_{ec}}^d + {f_{bc}}^e {f_{ea}}^d + {f_{ca}}^e {f_{eb}}^d = 0\,. \tag{C.41}$$

If we now regard the structure constants as elements of $n \times n$ matrices f_a according to

$$(f_a)^c{}_b \equiv f_{ab}{}^c \,, \tag{C.42}$$

we can rewrite (C.41) as a matrix identity

$$-(f_c)^d{}_e(f_a)^e{}_b + (f_a)^d{}_e(f_c)^e{}_b - f_{ac}{}^e(f_e)^d{}_b = 0 \,. \tag{C.43}$$

or, after relabeling of indices,

$$([f_a, f_b])^d{}_e = f_{ab}{}^c(f_c)^d{}_e \,. \tag{C.44}$$

Consequently the matrices f_a generate a representation of the Lie algebra and therefore of the Lie group; clearly, the adjoint representation has dimension n, like the Lie group itself. Obviously an abelian group has vanishing structure constants, so that its adjoint representation is trivial, i.e. it consists of the identity element.

As an example consider again SU(2). As shown above this group has three generators t_a and structure constants $f_{ab}{}^c$ equal to $-\varepsilon_{abc}$. Therefore the adjoint representation is 3-dimensional with generators S_a, given by

$$(S_a)^c{}_b = -\varepsilon_{abc} \,, \tag{C.45}$$

or, explicitly:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \,, \quad S_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \,, \quad S_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.46}$$

which are just the generators of SO(3). Hence SU(2) and SO(3) have the same structure constants and are therefore *locally* equivalent. This fact is of physical importance because rotations of spatial coordinates are governed by SO(3), while spin rotations are described by SU(2); hence spatial and spin rotations form different representations of the same group.

For higher-dimensional matrices it becomes more difficult to construct the finite group elements by explicit cxponentiation, but for SO(3) this is still feasible. We first calculate powers of $\xi^a S_a$:

$$\begin{aligned} (\xi^a S_a)^2 &= -\xi^2 \mathrm{I} + \Xi(\xi) \,, \\ (\xi^a S_a)^3 &= -\xi^2(\xi^a S_a) \,, \end{aligned} \tag{C.47}$$

where ξ is defined by (C.24) and the symmetric 3×3 matrix $\Xi(\xi)$ is defined by

$$\Xi(\xi)^{ab} = \xi^a \xi^b \,. \tag{C.48}$$

Generalizing this to

$$(\xi^a S_a)^{2n} = (-\xi^2)^n(\mathrm{I} - \xi^{-2}\Xi(\xi))\,, \quad (\xi^a S_a)^{2n+1} = (-\xi^2)^n(\xi^a S_a)\,, \tag{C.49}$$

we can straightforwardly exponentiate $\xi^a S_a$:

$$\begin{aligned} g_{\mathrm{SO}(3)}(\xi) &= \exp(\xi^a t_a) \\ &= \sum_{n=0}^{\infty}\Big\{\frac{1}{(2n)!}(-1)^n\xi^{2n}\mathrm{I} + \frac{1}{(2n+1)!}(-1)^n\xi^{2n}\xi^a S_a \\ &\qquad + \frac{1}{(2n+2)!}(-1)^n\xi^{2n}\Xi(\xi)\Big\} \\ &= \cos\xi\mathrm{I} + \frac{\sin\xi}{\xi}\xi^a S_a + \frac{1-\cos\xi}{\xi^2}\Xi(\xi)\,. \end{aligned} \tag{C.50}$$

Because of the periodicity of (C.50) in ξ, the parameter space can now be restricted to the inside of a 2-dimensional sphere with radius π, i.e.

$$(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \leqslant \pi^2\,. \tag{C.51}$$

It is illuminating to compare the parameter space of SO(3) to that of SU(2) (a slice of the latter is shown in fig. C.1). The parameter space of SU(2) covers the group SO(3) twice, because two points ξ and ξ' satisfying (C.27) correspond to the same element of SO(3). For this reason the parameter space of SO(3) can be restricted to the inner region with $\xi \leqslant \pi$. Opposite points on the 2-dimensional sphere with radius π that forms the boundary of this region correspond to the *same* SO(3) element in contradistinction with the corresponding SU(2) elements which differ by a sign (cf. C.28). This implies that there are paths that are closed in the SO(3) group manifold (but not closed in the SU(2) manifold), which cannot be continuously deformed to a point. Hence the SO(3) manifold is not simply connected.

Finite transformations in the adjoint representation can be obtained by exponentiation, just as in (C.50). Hence one has $n\times n$ transformation matrices defined by

$$g_{\mathrm{adj}}(\xi) = \exp(\xi^a f_a)\,. \tag{C.52}$$

Quantities transforming in this representation are n-dimensional vectors ϕ^a. A convenient way of dealing with such vectors is based on a matrix notation

$$\Phi = \phi^a t_a\,, \tag{C.53}$$

where t_a are the group generators in some arbitrary representation. The transformation

$$\Phi \to \Phi' = g\Phi g^{-1}\,, \tag{C.54}$$

with g in the same representation as t_a [so that $g = \exp(\xi^a t_a)$], now induces the same transformation on the ϕ^a as (C.52), i.e.

$$g(\phi^a t_a)g^{-1} = \phi'^a t_a \,, \tag{C.55a}$$

with

$$\phi'^a = (g_{\rm adj}\phi)^a \,. \tag{C.55b}$$

An important feature of this result is that (C.55a) can still be decomposed into the same matrices t_a for $g \neq \mathbf{I}$. This fact follows from the group axioms. To see this choose group elements g and g_1, and observe that the product

$$g_2 = g g_1 g^{-1} \tag{C.56}$$

is also an element of the group. If we now assume that g_1 is an infinitesimal transformation, i.e.

$$g_1 = \mathrm{I} + \phi^a t_a + O(\phi^2) \,, \tag{C.57}$$

and retain only terms of first order in ϕ on both sides of (C.56), then also g_2 must take the form of an infinitesimal transformation

$$g_2 = \mathrm{I} + \phi'^a t_a + O(\phi^2) \,, \tag{C.58}$$

with ϕ'^a linearly proportional to ϕ^a. Substituting (C.57) and (C.58) into (C.56) now gives (C.55a), so that it only remains to prove that the relation between ϕ'^a and ϕ^a is given by (C.55b). This can easily be done for finite transformations g (see problem C.2), but we will content ourselves with infinitesimal transformations g. Hence we assume

$$g = \mathrm{I} + \xi^a t_a + O(\xi^2) \,, \quad g^{-1} = \mathrm{I} - \xi^a t_a + O(\xi^2) \,,$$

so that the left-hand side of (C.55a) reads

$$\begin{aligned} g(\phi^a t_a)g^{-1} &= \phi^a t_a + [\xi^b t_b, \phi^c t_c] + O(\xi^2) \\ &= (\phi^a + f_{bc}{}^a \xi^b \phi^c) t_a + O(\xi^2) \,. \end{aligned}$$

Consequently

$$\phi'^a = (\delta^a_c + \xi^b f_{bc}{}^a + O(\xi^2))\phi^c \,, \tag{C.59}$$

which is just the result of an infinitesimal transformation acting on ϕ^a in the adjoint representation, thus confirming (C.55b).

One reason why the above matrix notation is so convenient is that one can easily construct invariants. For instance the trace over products of Φ's is manifestly invariant. The simplest example of this is

$$\mathrm{Tr}(\Phi_1\Phi_2) = g^{\mathrm{R}}_{ab}\phi^a_1\phi^b_2\,, \tag{C.60}$$

where g^{R}_{ab} is a group invariant "metric" tensor defined by

$$g^{\mathrm{R}}_{ab} = \mathrm{Tr}(t_a t_b)\,. \tag{C.61}$$

The superscript R indicates that it depends on the representation adopted for the t_a. Invariance of g^{R}_{ab} means that $g^{\mathrm{R}}_{ab}\phi^a\phi^b = g^{\mathrm{R}}_{ab}\phi'^a\phi'^b$ and follows from (C.55) and (C.60); for infinitesimal transformations this yields

$$\delta g^{\mathrm{R}}_{ab} \propto f_{ca}{}^{d} g^{\mathrm{R}}_{db} + f_{cb}{}^{d} g^{\mathrm{R}}_{ad} = 0\,. \tag{C.62}$$

Semisimple groups
Lie groups (and their corresponding algebras) can be subdivided in three main classes based on the presence or absence of invariant subgroups (or correspondingly invariant subalgebras). An *invariant subgroup H* satisfies

$$g \cdot h = h' \cdot g \tag{C.63}$$

for any element gG and $h, h'H$. For the generators corresponding to G and h, this implies that the only nonvanishing commutators involving the generators of H are

$$[t_g, t_h] = t'_h\,, \tag{C.64}$$

where t_g is an arbitrary generator of G and t_h and t'_h are linear combinations of the generators of H. Groups that have *no* invariant subgroups are called *simple* (obviously we exclude the two trivial invariant subgroups here: the group itself, and the identity element).

A weaker restriction is that the group has no *abelian* invariant subgroups; such groups are called *semisimple* (the group U(1) is an exception; although it has no nontrivial subgroups it is called nonsemisimple). Semisimple groups are thus allowed to have nonabelian invariant subgroups. However, in that case one can show that the group *factorizes*, i.e. it can be written as a *direct product* of simple groups

$$G = G_1 \times G_2 \times \cdots\,, \tag{C.65}$$

where $G_1, G_2, \ldots$, are simple (nonabelian) groups which are mutually commuting: elements $g_iG_i, g_jG_j (i \neq j)$ commute:

$$g_i \cdot g_j = g_j \cdot g_i\,. \tag{C.66}$$

Correspondingly the Lie algebra of a semisimple group can be decomposed into simple nonabelian algebras. A well-known example of a semisimple group is SO(4), the group of 4-dimensional real rotations, which factorizes (locally) according to

$$\mathrm{SO}(4) = \mathrm{SO}(3) \times \mathrm{SO}(3)\,. \tag{C.67}$$

Finally, groups that contain abelian invariant subgroups are called *nonsemisimple*. As we shall demonstrate shortly, such groups do not always factorize, i.e. they cannot always be written as the direct product of an abelian group and a semisimple group.

As an example consider the 3-parameter Lie groups SU(2), Sl(2, R) and E_2 (the latter is the Euclidean group, consisting of rotations and translations in a 2-dimensional plane; t_l and t_2 will denote the generators of the two translations and t_3 the generator of the rotations). The non-vanishing structure constants for these groups have been collected in table C.1. The first

Table C.1
Nonvanishing structure constants for SU(2), Sl(2, R) and E_2.

Group	${f_{23}}^1$	${f_{31}}^2$	${f_{12}}^3$
SU(2)	-1	-1	-1
Sl(2,R)	1	-1	1
E_2	-1	-1	0

two groups are clearly simple; although they have one-parameter (abelian) subgroups, those are not invariant (to see this, write the generator of the subgroup as $\alpha^a t_a$ and impose the condition (C.64) which requires that $[\alpha^a t_a, t_b]$ must be proportional to $\alpha^a t_a$ for all t_b; this yields $\alpha^a = 0$). The group E_2 has an obvious two-parameter abelian subgroup consisting of translations in the plane; under a rotation R a translation T is converted into another translation T′, i.e.

$$\mathrm{RTR}^{-1} = \mathrm{T}'\,.$$

This is just the condition (C.63) so that the translations constitute an invariant abelian subgroup of E_2. Consequently E_2 is *nonsemisimple*.

An important quantity is the so-called Cartan-Killing metric, which is a special case of (C.61) in the adjoint representation. It is defined by

$$g_{ab} = {f_{ad}}^c {f_{bc}}^d\,. \tag{C.68}$$

As shown by Cartan the metric (C.68) is nonsingular (i.e. det $(g \neq 0)$ if and only if the group is semisimple. To verify this result for the groups SU(2), Sl(2, R) and E_2 is straightforward. Using the structure constants of table C.1, one finds

$$g_{ab} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{for} \quad \mathrm{SU}(2)\,, \tag{C.69a}$$

$$g_{ab} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{for} \quad \mathrm{Sl}(2,\mathrm{R})\,, \tag{C.69b}$$

$$g_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{for} \quad \mathrm{E}_2\,. \tag{C.69c}$$

The presence of zero eigenvalues in the third formula thus confirms that E_2 is a nonsemisimple group.

The fact that the metric is a symmetric invariant tensor implies that (C.62) must be satisfied. Using the metric to lower the index in the structure constants according to

$$f_{abc} = f_{ab}{}^{d} g_{dc}\,, \tag{C.70}$$

the condition (C.62) together with the antisymmetry of the $f_{ab}{}^{c}$ in the lower indices implies that f_{abc} is totally antisymmetric,

$$f_{abc} = -f_{bac} = -f_{cba}\,. \tag{C.71}$$

Combining the results of table C.1 and (C.69) the antisymmetry can be verified for the groups SU(2), Sl(2, R) and E_2. Note that for *any* choice of the invariant metric one obtains an antisymmetric tensor f_{abc}.

The Cartan-Killing metric is a real symmetric matrix, which can therefore be diagonalized by means of an orthogonal redefinition of the generators. Adopting suitable normalization constants for the generators it is possible to write the metric as a diagonal matrix with eigenvalues $+1, -1$, or 0. If there are zero eigenvalues as in (C.69c) the group is nonsemisimple. If the metric is negative definite, as in (C.69a), the group is compact. A metric with both positive and negative eigenvalues is noncompact, as is demonstrated by (C.69b). Note that metrics defined in different representations need not be proportional to one another. We return to this question at the end of this section.

Since the Cartan-Killing metric is nonsingular for semisimple groups it is possible to define an inverse g^{ab}:

$$g^{ab} g_{bc} = \delta^{a}{}_{c}\,. \tag{C.72}$$

The inverse metric can be used to define a matrix,

$$C = g^{ab} t_a t_b \,, \tag{C.73}$$

called the Casimir operator, which is invariant in any representation, i.e.

$$g C g^{-1} = C \,, \tag{C.74}$$

where g denotes any group element in the representation corresponding to t_a. To verify (C.74), consider an infinitesimal transformation, for which

$$\delta C \propto [t_a, C] \,.$$

Substituting (C.73) one has

$$\begin{aligned}
[t_a, C] &= g^{bc}(t_a t_b t_c - t_b t_c t_a) \\
&= g^{bc}([t_a, t_b] t_c + t_b [t_a, t_c]) \\
&= f_{ab}{}^{d} g^{bc} t_d t_c + f_{ac}{}^{d} g^{bc} t_b t_d \\
&= f_{abe} g^{ed} g^{bc} t_d t_c + f_{ace} g^{ed} g^{bc} t_b t_d \,,
\end{aligned} \tag{C.75}$$

where we have used the inverse of (C.70),

$$f_{ab}{}^{c} = g^{ce} f_{abe} \,. \tag{C.76}$$

Due to the antisymmetry of f_{abc} the two terms in (C.75) cancel, so that C is indeed invariant.

The type of invariant matrices is restricted by Schur's lemma:

If a matrix commutes with every element in an *irreducible* representation of a group, then this matrix must be proportional to the identity. For Lie groups the lemma may also be rephrased as: if a matrix commutes with all generators of the group in an *irreducible* representation then it must be proportional to the identity.

On the basis of this lemma one thus concludes that, in an irreducible representation labelled by R,

$$C(\mathrm{R}) = d_{\mathrm{R}} \mathrm{I} \,. \tag{C.77}$$

where d_{R} is a number depending on the representation. For the adjoint representation, it follows that $d_{\mathrm{R}} = 1$, i.e.

$$(C(\text{adjoint}))^{c}_{d} = f_{ae}{}^{c} f_{bd}{}^{e} g^{ab} = \delta^{c}_{d} \,. \tag{C.78}$$

For the defining representations of SU(2) and Sl(2, R), given in (C.21) and in (C.30), we have

$$C = \tfrac{3}{8} \mathrm{I} \quad \text{for} \quad \mathrm{SU}(2) \quad \text{and} \quad \mathrm{Sl}(2, \mathrm{R}) \,. \tag{C.79}$$

For the rotation group SO(3) the Casimir operator is just the total angular momentum operator (modulo a factor $2\hbar^2$), and one has

$$C = \tfrac{1}{2}l(l+1)\mathrm{I} \tag{C.80}$$

where l labels the representations.

Schur's lemma can be used to derive another useful result concerning invariant metrics defined in different representations. Using the Cartan Killing metric to raise and lower indices, it is possible to rewrite (C.62) as

$$f_{ce}{}^{a}(g^{-1}g^{\mathrm{R}})^{e}{}_{b} - (g^{-1}g^{\mathrm{R}})^{a}{}_{e}f_{cb}{}^{e} = 0\,, \tag{C.81}$$

where $g^{-1}g^{\mathrm{R}}$ is the inverse Cartan-Killing metric times the metric defined in (C.61) for a representation R. In deriving (C.81) we made use of the anti-symmetry property (C.71). According to (C.81) the matrix $g^{-1}g^{\mathrm{R}}$ commutes with all the generators in the adjoint representation. For a simple group this representation is irreducible, so that Schur's lemma implies $g^{-1}g^{\mathrm{R}} \propto \mathrm{I}$, or

$$g^{\mathrm{R}}_{ab} = c_{\mathrm{R}}\, g_{ab}\,. \tag{C.82}$$

For a semisimple group one must decompose the metric according to the various representations, for each of which one has a proportionality relation. However, as the proportionality constants may differ from one irreducible representation to another, (C.82) does not necessarily hold.

The proportionality constants c_R and d_R are related. This follows from contracting (C.82) with g^{ab} and using (C.61) and (C.77), which leads to

$$c_{\mathrm{R}} = \frac{\dim R}{\dim G}\, d_{\mathrm{R}}\,. \tag{C.83}$$

Here dim R and dim G are the dimensions of the representation R and of the group; the latter coincides with the dimension of the adjoint representation. Note that in this representation $c_{\mathrm{R}} = d_{\mathrm{R}} = 1$. Applying (C.83) to the defining representations of SU(2) and Sl(2,R) yields

$$c_R = \tfrac{2}{3}\cdot\tfrac{3}{8}\,, \tag{C.84}$$

where we use that $d_{\mathrm{R}} = 3/8$ according to (C.79). Consequently, the metric equals $\frac{1}{4}$ times the Cartan-Killing metric, a result than can be verified directly from the generators defined in (C.21) and (C.30).

Problems

10.1. Show that a general SU(2) transformation can be written as

$$g = \alpha_0 \mathrm{I} + \mathrm{i}\alpha^a\tau_a \quad \text{with} \quad (\alpha^0)^2 + (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 = 1\,.$$

Argue that the SU(2) parameter space is the inside of a three-dimensional sphere.

10.2. Prove that

$$\exp(-\xi^b t_b) t_a \exp(\xi^c t_c) = t_b [\exp(-\xi^c f_c)]^b{}_a \,, \tag{1}$$

where $(f_c)^b{}_a$ is the group generator in the adjoint representation (c.f C.42) by scaling $\xi \to \lambda\xi$ and proving that both sides satisfy the same linear differential equation in λ i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} X(\lambda) = [X(\lambda), \xi^a t_a] \,, \tag{2}$$

where $X(\lambda)$ is the modifed left- or right-hand side of (1). Fix the proportionality constant by comparing the left-and right-hand sides for $\lambda = 0$. Observe that an example of this relation was derived in problem 5.1.

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