

REVIEW PART 2: Calculation of QED Cross Sections and Decay Rates

Overview:

- Continue formulas for cross section and decay calculations
- Cross section calculation for spinless muon-electron
- Including spin (Dirac equation refresher)

(This lecture mostly follows Halzen and Martin Chap. 4-6 and Griffiths Chap. 6-7)

Differential cross section (applications) ②

Last lecture we obtained:

$$d\sigma = \frac{|M|^2}{F} dQ \rightarrow \text{invariant amplitude}$$

↳ Lorentz invariant phase space factor (dLips)

↳ Incident Flux

- we obtained an expression for M : (cm)

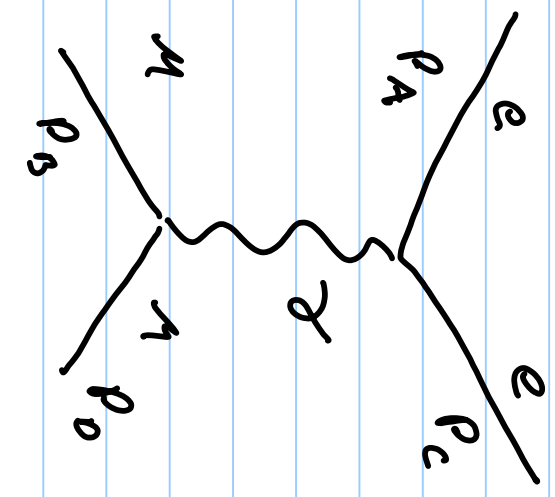
$$-iM = \text{ie} (p_A + p_C)^{\mu} \left(-\frac{ig_{\mu\nu}}{q^2} \right) \text{ie} (p_B + p_D)^{\nu}$$

- we will now calculate dQ and F for the case where we chose the centre of mass frame i.e.

Find dQ, F for AB \rightarrow CD in CM

CM Frame, eM scattering

$$\begin{aligned}
 dQ &= (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D} \\
 &= \frac{1}{4\pi^2} \frac{d^3 p_C}{2E_C} \frac{d^3 p_D}{2E_D} \delta^4(p_A + p_B - p_C - p_D) \\
 &= \frac{1}{4\pi^2} \frac{d^3 p_C}{2E_C} \frac{d^3 p_D}{2E_D} \delta(E_A + E_B - E_C - E_D) \delta^3(-p_C - p_D) \\
 &= \frac{1}{4\pi^2} \frac{d^3 p_C}{2E_C 2E_D} \delta(W - E_C - E_D) \rightarrow \text{we integrated over } p_D
 \end{aligned}$$



$$\vec{p}_A = -\vec{p}_B$$

- p_A : 4-vector
- \vec{p}_A : 3-vector
- p_A : scalar

CM Frame, em scattering (cont) (4)

$$d^3 p_c^{\vec{s}} = d^3 p_c^{\vec{2}} = p_c^2 dp_c \underbrace{\sin\theta d\theta d\varphi}_{d\Omega}$$

so we have:

$$dQ = \frac{1}{4\pi^2} \frac{p_c^2 d\Omega dp_c \delta(W - E_3 - E_4)}{4E_c E_0}$$

$$W = E_c + E_0 = \sqrt{m_c^2 + p_c^2} + \sqrt{m_0^2 + p_c^2}$$

$$\frac{dW}{dp_c} = \frac{p_c}{E_c} + \frac{p_c}{E_0}$$

$$dp_c = \frac{dW}{p_c} \frac{E_c E_0}{(E_c + E_0)}$$

CM Frame, em scattering (cont) (5)

we now have:

$$dQ = \frac{1}{4\pi^2} \frac{\rho_E}{4} \left(\frac{1}{E_c + E_0} \right) dW d\Omega \delta(W - E_c - E_b)$$
$$= \boxed{\frac{1}{4\pi^2} \frac{\rho_E}{4\sqrt{s}} d\Omega}$$

Now we find F:

From last lecture we had $F = |v_A|^2 2E_A 2E_B$

$$v_A \vec{s} = \frac{\vec{p}_A}{E_A}, \quad \text{for a collinear collision}$$

$$F = |v_A \vec{s} - v_B \vec{s}| \cdot 2E_A \cdot 2E_B = |v_A \vec{s} - v_B \vec{s}| \cdot (|v_A \vec{s}| + |v_B \vec{s}|)$$

$$= \left(|v_A \vec{s}| + |v_B \vec{s}| \right) \cdot 2E_A \cdot 2E_B$$

$$= 4 \left(|p_A| E_B + |p_B| E_A \right) = \boxed{4 p_i \cdot \sqrt{s}}$$

CM Frame, e μ scattering (cont) ⑥

We then get:
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{p_f}{p_i} |M|^2$$

— Now, we evaluate $|M|^2$ [neglecting masses]

we have $p_A = (p, \vec{p}_i)$, $p_B = (p, -\vec{p}_i)$

$$p_C = (p, \vec{p}_f), \quad p_D = (p, -\vec{p}_f)$$

$$-iM = (ie(p_A + p_C)^\mu) \left(\frac{-ig_{\mu\nu}}{q^2} \right) (ie(p_B + p_D)^\nu)$$

$\rightarrow q^2 = (p_D - p_B)^2$

$$(p_A + p_C) = (2p, \vec{p}_i + \vec{p}_f)$$

$$(p_B + p_D) = (2p, -(\vec{p}_i + \vec{p}_f))$$

$$(p_D - p_B) = (0, -\vec{p}_f + \vec{p}_i)$$

CM Frame, e μ scattering (cont) (7)

$$q^2 = (P_D - P_N)^2 = (0, \vec{p}_i - \vec{p}_f)^2$$

$$= 0 - (\vec{p}_i - \vec{p}_f)^2$$

$$= - [p_i^2 + p_f^2 - 2pp_i \cos \theta]$$

$$= - [2p^2 - 2p^2 \cos \theta] = - (2p^2 (1 - \cos \theta))$$

$$(P_A + P_C)^{\mu} (P_B + P_D)^{\mu} =$$

$$= 4p^2 - [(\vec{p}_i + \vec{p}_e) \cdot (-\vec{p}_i - \vec{p}_f)]$$

$$\rightarrow = -p^2 - p^2 - p^2 \cos \theta - p^2 \cos \theta$$

$$= 6p^2 + 2p^2 \cos \theta$$

$$= 2p^2 (3 + \cos \theta)$$

CM Frame, e μ scattering (cont)

(8)

we had $-iM = \dots ie \cdot -i \cdot ie \dots$

$$M = i \cdot ie \cdot -i \cdot ie \dots = -e^2 \dots$$

$$M = \frac{-e^2 \cdot 2p^2 (3 + \cos\theta)}{-2p^2 (1 - \cos\theta)}$$

$$= e^2 \frac{(3 + \cos\theta)}{(1 - \cos\theta)}$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{(4\pi)^2} \frac{p_f}{p_i} \cdot \frac{1}{s} \cdot \frac{(3 + \cos\theta)^2}{(1 - \cos\theta)^2}$$

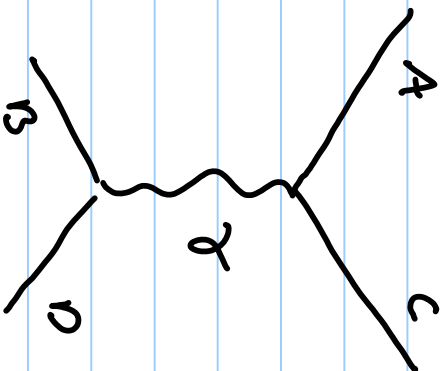
$$\alpha = \frac{e^2}{4\pi}$$
$$\alpha^2 = \frac{e^4}{16\pi^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \cdot \frac{(3 + \cos\theta)^2}{(1 - \cos\theta)^2}$$

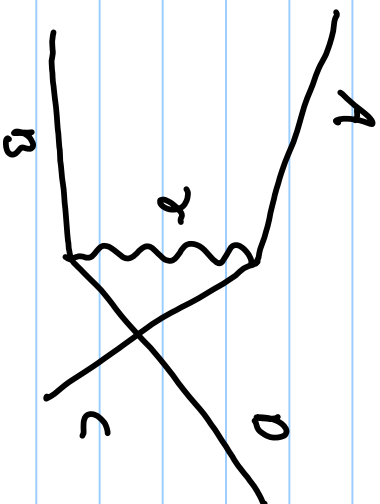
Spinless electron-electron scattering

(9)

we now have 2 diagrams:



+



giving us

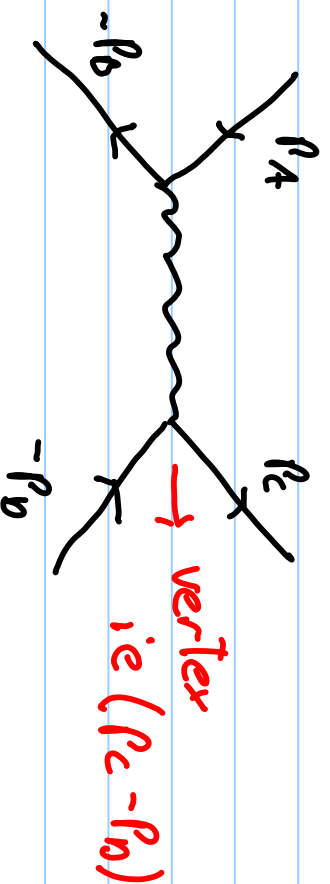
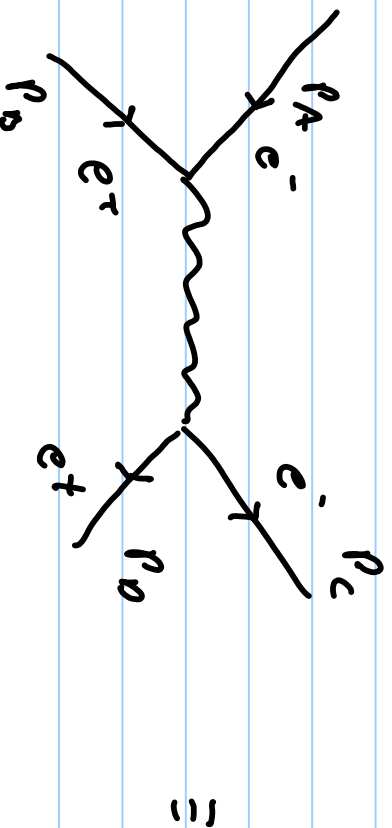
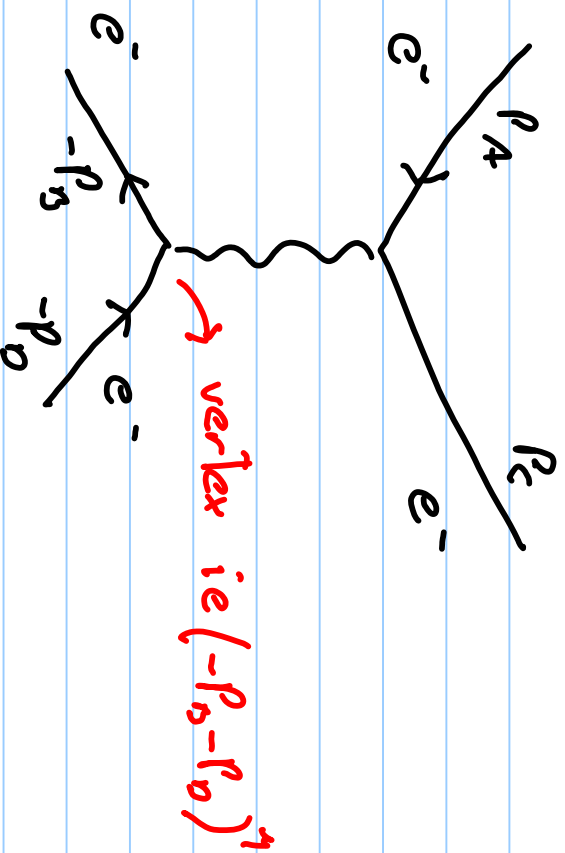
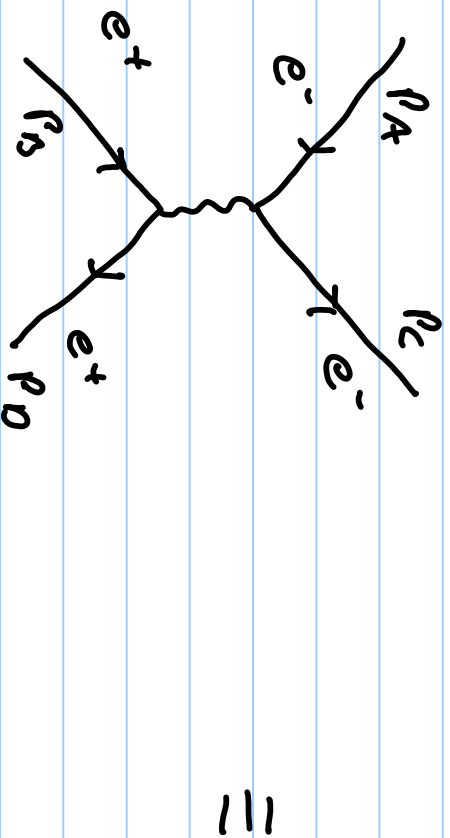
$$\sim iM = -i \left(-e^2 \frac{(p_A + p_C)_\mu (p_B + p_D)_\mu}{(p_B - p_A)^2} - e^2 \frac{(p_A + p_D)_\mu (p_B + p_C)_\mu}{(p_C - p_B)^2} \right)$$

what we had
for e_m scattering

Electron - Positron scattering

(10)

we have 2 diagrams:



$$-iM_{e^+e^-} = -i \left[-e^2 (p_A + p_C)_\mu (-p_B - p_D)^\mu - e^2 (p_A - p_B)_\mu (-p_D + p_C)^\mu \right] e^+ e^-$$

Invariant variables (Mandelstam variables)

(11)

$$S = (P_A + P_B)^2$$

$$T = (P_A - P_C)^2$$

$$U = (P_A - P_B)^2$$

For $e^-e^+ \rightarrow e^-e^+$ in CM we get

$$P_A = (E, \vec{p}_i), \quad P_B = (E, -\vec{p}_i)$$

$$P_A + P_B = (2E, 0)$$

$$(P_A + P_B)^2 = 4E^2 = 4(p^2 + m^2) = S$$

$\hookrightarrow M_{e^-} = M_{e^+}$

$T?$

$$P_C = (E, \vec{p}_f)$$

$$(P_A - P_C)^2 = (0, \vec{p}_i - \vec{p}_f)$$

$$(P_A - P_C)^2 = -\left[(\vec{p}_i - \vec{p}_f)^2 \right]$$

Invariant variables (Mandelstam variables) cont. (12)

$$T = - \left[p_i^2 + p_f^2 - 2 p_i p_f \cos \theta \right]$$

$$= - \left[2 p^2 (1 - \cos \theta) \right]$$

$p_i = p_f$ in CM

$$U = -2 p^2 (1 + \cos \theta)$$

We can rewrite $M_{e^-e^-}$ in terms of Mandelstam variables:

$$M_{e^-e^-} = e^2 \left(\frac{u-s}{t} + \frac{t-s}{u} \right)$$

$$M_{e^+e^-} = e^2 \left(\frac{s-u}{t} + \frac{t-u}{s} \right)$$

Decay Rates

(13)

$$A \rightarrow 1 + 2 + \dots + N$$

$$d\Gamma = \frac{1}{2E_A} |M|^2 \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \dots \frac{d^3 p_N}{(2\pi)^3 2E_N} \delta^4(P_A - p_1 - p_2 - \dots - p_N)$$

For particle A decaying into particles 1 and 2:

$$\Gamma = \frac{1}{M_A 32\pi^2} \int \frac{|M|^2}{E_1 E_2} \delta^4(P_A - p_1 - p_2) d^3 p_1 d^3 p_2$$

- Example: Find Γ for the process $\pi^0 \rightarrow \gamma\gamma$ if the invariant amplitude is M .

$$\delta^4(P_A - p_1 - p_2) = \delta(M_A - E_1 - E_2) \delta^3(-\vec{p}_1 - \vec{p}_2)$$

$$M_1 = M_2 = 0 \Rightarrow E_1 = p_1 \text{ and } E_2 = p_2$$

$$\rightarrow \Gamma = \frac{1}{32\pi^2 M_A} \int \frac{|M|^2}{p_1 p_2} \delta(M_A - p_1 - p_2) \delta^3(-\vec{p}_1 - \vec{p}_2) d^3 p_1 d^3 p_2$$

Example continued

(14)

$$P = \frac{|M|^2}{32\pi^2 M_A} \int |M|^2 \delta(M_A - 2p_1) d^3\vec{p}_1$$

$$d^3\vec{p}_1 = p_1^2 \sin\theta d\theta d\varphi, \quad \int \sin\theta d\theta d\varphi = 4\pi$$

so:

$$P = \frac{|M|^2}{8\pi M_A} \int_0^\infty \delta(M_A - 2p_1) dp_1$$

$$\delta(M_A - 2p_1) = \frac{1}{2} \delta\left(p_1 - \frac{M_A}{2}\right)$$

$$P = \frac{1}{16\pi M_A} |M|^2$$

Cross Section for e μ scattering (15)

Adding spin \rightarrow Dirac Equation

START FROM:

$$H\psi = (2 \cdot \vec{p} + \beta m)\psi \quad \text{①}$$

\hookrightarrow linear in $\frac{d}{dt}$, ∇

and: $H^2\psi = (p^2 + m^2)\psi \quad \text{②}$

\hookrightarrow follows $E^2 = p^2 + m^2$

Dirac-Pauli REP.

$$\gamma^\mu \equiv (\beta, \beta\alpha) \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Equation (cont)

(16)

We want to find eigenvectors for (1)

$$Hv = (\alpha \cdot \vec{p} + \beta m)v = Ev$$

4 solutions: Two with $E > 0$
Two with $E < 0$

First look at particle at rest:

$$p=0 \quad \rightarrow \quad Hv = \beta mv = \begin{pmatrix} mI & 0 \\ 0 & mI \end{pmatrix} v$$

Eigenvalues: $m, -m$

$E < 0$ solutions are
To be interpreted as
 $E > 0$ solutions for
anti-particles

Eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Dirac Equation (cont.)

(17)

We now consider the case where $p \neq 0$

$$Hv = \begin{pmatrix} m & \sigma \cdot \vec{p} \\ \sigma \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = E \begin{pmatrix} v_A \\ v_B \end{pmatrix}$$

$$\sigma \cdot \vec{p} v_B = (E - m) v_A \quad (3)$$

$$\sigma \cdot \vec{p} v_A = (E + m) v_B \quad (4)$$

First, for $E > 0$ solutions, we set $v_A^{(s)} = \chi^{(s)}$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{From (4)} \quad v_B^{(s)} = \frac{\sigma \cdot \vec{p}}{E + m} \chi^{(s)}$$

$$\rightarrow v^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \frac{\sigma \cdot \vec{p}}{E + m} \chi^{(s)} \end{pmatrix} \quad E > 0, \quad s = 1, 2$$

$$N = \sqrt{E + m}$$

Dirac Equation (cont.)

(18)

we get for $E < 0$, $s=3,4$

$$u^{(s)} = N \begin{pmatrix} -\frac{\sigma \cdot \vec{p}}{|E|+m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

Completeness relations

$$\sum_{s=1,2} u^{(s)}(p) \bar{u}^{(s)}(p) = \not{p} + m$$
$$\sum_{s=1,2} v^{(s)}(p) \bar{v}^{(s)}(p) = \not{p} - m$$

\hookrightarrow pos. Tron

From (1) we get

$$i \gamma^{\mu} \partial_{\mu} \not{\psi} - m \not{\psi} = 0 \quad (3)$$

$$i \partial_{\mu} \bar{\not{\psi}} \gamma^{\mu} + m \bar{\not{\psi}} = 0 \quad (4)$$

Hermitian conjugate

Some useful relations:

$$\bar{\not{\psi}} \equiv \not{\psi}^{\dagger} \gamma^0$$

$$\gamma^{0\dagger} = \gamma^0, \quad (\gamma^0)^2 = I, \quad \gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu} = 2 g^{\mu\nu}$$

$$\gamma^{k\dagger} = -\gamma^k, \quad \gamma^{m\dagger} = \gamma^0 \gamma^m \gamma^0$$

$$\not{\psi} \equiv \gamma^{\mu} p_{\mu}$$

Dirac Equation (cont.)

(19)

Other results: $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$, $\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma^{5+} = \gamma^5, \quad (\gamma^5)^2 = I$$
$$\gamma^5\gamma^m + \gamma^m\gamma^5 = 0$$

We now find an expression for the current density:

TAKE (3) $\times \bar{\psi}$ FROM THE LEFT AND
TAKE (4) $\times \psi$ FROM THE RIGHT:

$$\bar{\psi}\gamma^\mu\partial_\mu\psi + (\partial_\mu\bar{\psi})\gamma^\mu\psi = \partial_\mu \underbrace{(\bar{\psi}\gamma^\mu\psi)}_{j^\mu} = 0$$

in fact we need to identify j^μ with the charge current density.

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi$$

Cross Section Calculations (adding spin) (20)

Electron with 4-momentum p^m :

$$\psi = u(\hat{p}) e^{i p \cdot x} \quad \text{will satisfy } (\gamma_\mu p^\mu - m)\psi = 0$$

we substitute: $p^m \rightarrow p^m + e A^m$

$$(\gamma_\mu p^\mu - m)\psi = \underbrace{\gamma^\nu V}_\psi \psi$$

$$\hookrightarrow = -e \gamma_\mu A^\mu$$

We go back to our Transition amplitude:

$$\begin{aligned} T_{fi} &= -i \int \psi_f^\dagger(x) V(x) \psi_i(x) d^4x \\ &= ie \int \bar{\psi}_f \gamma_\mu A^\mu \psi_i d^4x \rightarrow \\ &= -i \int j_\mu^{Fi} A^\mu d^4x \end{aligned}$$

$$\begin{aligned} j_\mu &= -e \bar{\psi}_f \gamma_\mu \psi_i \\ j_\mu &= -e \bar{u}_f \gamma_\mu u_i e^{i(p_f - p_i) \cdot x} \end{aligned}$$

Note: now vertex factor has $\psi \psi$

Note: now interaction involves magnetic moment of electron

