LECTURE 18: QCD (Part 1)

Overview:

- QCD introduction
- Feynman rules
- Calculating colour factors
- quark anti-quark annihilation

(I used Quigg and mostly Giffiths as references)

QCD (I)

Evidence that quarks come in 3 different "colours" include

- → spin-stat. problem for baryons
- → cross section $e^-e^+ \to$ hadrons
- → Υ branching ratios
- → π^0 lifetime
- → anomaly cancelation

Could we use $U(3)$, $SO(3)$ as candidate gauge groups for the strong interaction?

$SO(3)$ → no asymptotic freedom

→ existence of diquark states, no distinction between quarks and anti-quarks

QCD (II)

$U(1)$ we get a singlet gauge boson $\rightarrow$ long-range interaction.

For the Lagrangian :

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})$$

$$\psi = \begin{pmatrix} q \\ q \\ q \end{pmatrix} \qquad D_\mu = \partial_\mu + igB_\mu$$

$\hookrightarrow$ 3x3 matrix : $B_\mu = \frac{1}{2}\lambda \cdot b_\mu = \frac{1}{2}\lambda^\ell b_\mu^\ell$

b_μ : 8 colour gauge fields

$$G_{\mu\nu} = \frac{1}{2} G_{\mu\nu} \cdot \lambda = \frac{1}{2} G_{\mu\nu}^\ell \lambda^\ell$$

$$= (ig)^{-1}[D_\nu, D_\mu] = \partial_\nu B_\mu - \partial_\mu B_\nu + ig[B_\nu, B_\mu]$$

QCD (III)

Properties of the λ matrices:

$\mathrm{Tr}(\lambda^l) = 0$ $\qquad\qquad$ $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$

$\mathrm{Tr}(\lambda^k \lambda^l) = 2\delta^{kl}$

$[\lambda^j, \lambda^k] = 2i f^{jkl} \lambda^l$ $\qquad\qquad$ $f_{147} = f_{246} = f_{345} = f_{516} = f_{637} = 1/2$

$$G^l_{\mu\nu} = \partial_\nu b^l_\mu - \partial_\mu b^l_\nu + g f^{jkl} b^j_\mu b^k_\nu$$

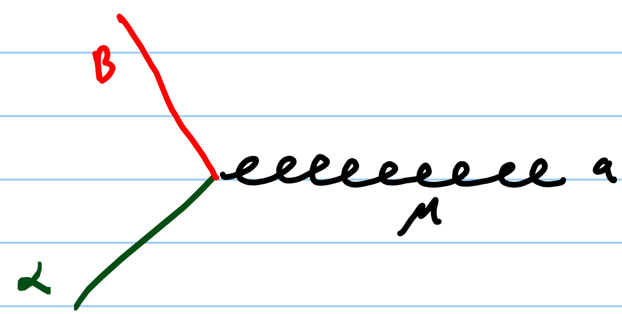
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

QCD IV

quark - gluon interaction: $\mathcal{L}_{int} = -\frac{g}{2} b^a_\mu \bar{\psi} \gamma^\mu \lambda^a \psi$



$-\frac{ig}{2} \lambda^a_{\alpha\beta} \gamma_\mu$

$\rightarrow$ one gluon exchange force prop. $\frac{g^2}{4} \sum_a \lambda^a_{\alpha\beta} \lambda^a_{\gamma\delta}$

$\alpha + \gamma \rightarrow \beta + \delta$

Gluons are massless particles with spin 1 which we will represent by pol. vector ε^μ.

$\rightarrow \quad \varepsilon^\mu p_\mu = 0$ (Lorentz Condition)

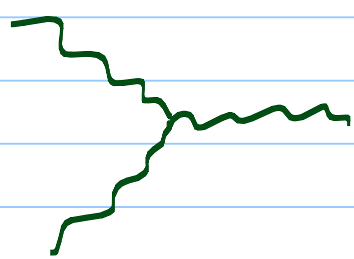
QCD (V)

We will also use Coulomb gauge: $\varepsilon^0 = 0 \qquad \vec{\varepsilon}\cdot\vec{p} = 0$

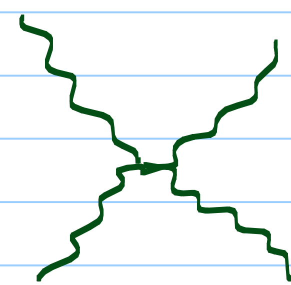
To describe colour state of gluon, we'll use column vector:

$$a = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \equiv |1\rangle \qquad \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \equiv |4\rangle$$

Non-Abelian group $\rightarrow$ gauge mediators interact with themselves:



3-gluon vertex

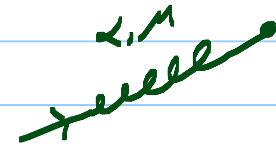


4-gluon vertex

QCD (VI)

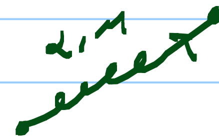
Feynman Rules

incoming gluon:



$\varepsilon_\mu(p)\, a^\alpha$

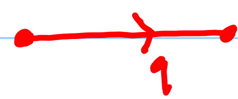
outgoing gluon



$\varepsilon^*_\mu(p)\, a^{\alpha *}$

Propagators

$q\bar{q}$

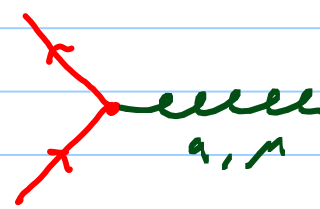


$$\frac{i(\not{q}+m)}{q^2-m^2}$$

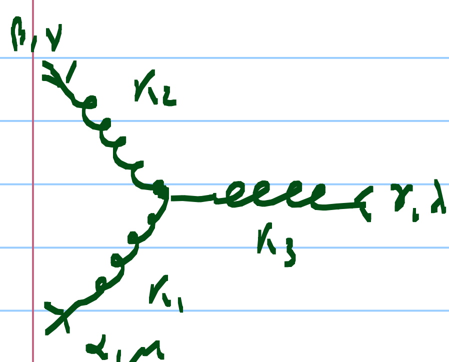
gluon

$$\frac{-i g_{\mu\nu}\, \delta^{\alpha\beta}}{q^2}$$

Vertices



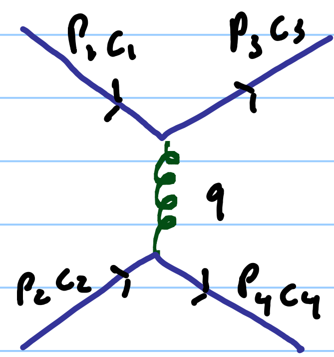
$$\frac{-ig_s}{2}\, \lambda^\alpha \gamma^\mu$$



$$-g_s f^{\alpha\beta\gamma}\left[g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_3-k_1)_\nu \right]$$

QCD

Consider $u + \bar{d} \rightarrow u + \bar{d}$



$$-i\mathcal{M} = \left[\bar{u}(3)\, c_3^{T}\right]\left[-i\frac{g_s}{2}\lambda^{\alpha}\gamma^{\mu}\right]\left[u(1)\, c_1\right]\left[\frac{-i g_{\mu\nu}\,\delta^{\alpha\beta}}{q^2}\right]$$

$$\times\left[\bar{v}(2)\, c_2^{+}\right]\left[-i\frac{g_s}{2}\lambda^{\beta}\gamma^{\nu}\right]\left[v(4)\, c_4\right]$$

$$\mathcal{M} = -\frac{g_s^2}{4}\frac{1}{q^2}\left[\bar{u}(3)\gamma^{\mu}u(1)\right]\left[\bar{v}(2)\gamma_{\mu}v(4)\right]\underbrace{(c_3^{T}\lambda^{\alpha}c_1)(c_2^{+}\lambda^{\alpha}c_4)}_{\text{colour factor}}$$

$\rightarrow$ same as electron-positron scattering with $g_e \rightarrow g_s$ but with extra colour factor

Potential similar to EM case $\quad V \propto \frac{\boxed{-F}\,\alpha_s}{r}$

QCD

- we'll consider the colour octet case e.g. $r\bar{b}$

$$c_1 = c_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad c_2 = c_4 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$F = \frac{1}{4}\left[(1\,0\,0)\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right] = \frac{1}{4}\lambda^\alpha_{11}\lambda^\alpha_{22}$$

$\rightarrow \lambda^3$ and λ^8 are the only relevant matrices

$$f = \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{11}\right) = -1/6$$

- now we consider the singlet state: $\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})$

$$f = \frac{1}{4}\cdot\frac{1}{\sqrt{3}}\Bigg[\left(c_3^\dagger \lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\right)[(1\,0\,0)\,\lambda^\alpha c_4] + \left(c_3^\dagger \lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\right)[(0\,1\,0)\lambda^\alpha c_4]$$
$$+ \left(c_3^\dagger \lambda^\alpha \begin{pmatrix}0\\0\\1\end{pmatrix}\right)[(0,0,1)\,\lambda^\alpha c_4]\Bigg]$$

QCD

outgoing state also in colour singlet state

→ we get 9 terms

$$f = \frac{1}{4} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \left(\lambda^{\alpha}_{ij} \lambda^{\alpha}_{ji} \right) = \frac{1}{12} \, \mathrm{Tr}\left(\lambda^{\alpha} \lambda^{\alpha} \right)$$

→ $\mathrm{Tr}\left(\lambda^{\alpha} \lambda^{\beta} \right) = 2 \delta^{\alpha\beta}$

→ $\mathrm{Tr}\left(\lambda^{\alpha} \lambda^{\alpha} \right) = 16$

$\Rightarrow f = +\frac{4}{3}$

→ $V_{q\bar{q}}(r) = \underbrace{-\frac{4}{3} \frac{\alpha_s}{r}}_{\text{singlet}}$, $V_{q\bar{q}}(r) = \underbrace{\frac{1}{6} \frac{\alpha_s}{r}}_{\text{octet}}$

QCD

quark - quark interaction

$$\mathcal{M} = -\frac{g_s^2}{4}\frac{1}{q^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right](c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\alpha c_2)$$

$$f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\alpha c_2)$$

we get a triplet:

$$\left.\begin{matrix}(rb - br)/\sqrt{2}\\(bg - gb)/\sqrt{2}\\(gr - rg)/\sqrt{2}\end{matrix}\right\}\ \text{anti-sym combinations}$$

and a sextet:

rr, gg, bb

+ symmetric combinations

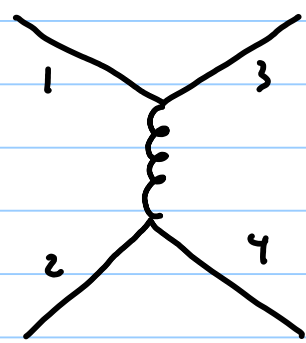
QCD

sextet : $c_1 = c_2 = c_3 = c_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (for rr)

$$f = \frac{1}{4}\left[(1\,0\,0)\, \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[(1\,0\,0)\, \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = \frac{1}{4} \lambda^\alpha_{11} \lambda^\alpha_{11}$$

$$= \frac{1}{4}\left[\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11} \right] = \frac{1}{4}\left[1 + \frac{1}{3} \right] = \frac{1}{3}$$

For Triplet: for example $(rb - br)/\sqrt{2}$



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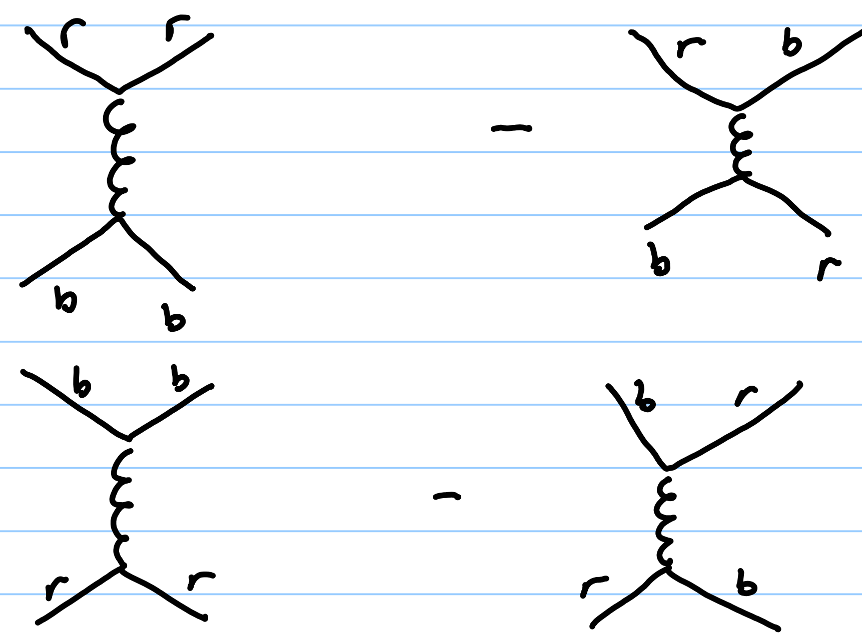


QCD

We get:

$$F = \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\Big[\Big[(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\Big]\Big[(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\Big]$$

$$- \Big[(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\Big]\Big[(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\Big]$$

$$+ \Big[(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\Big]\Big[(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\Big]$$

$$- \Big[(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}\Big]\Big[(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\Big]\Big]$$

$$= \frac{1}{8}\left[\lambda^\alpha_{11}\lambda^\alpha_{22} - \lambda^\alpha_{21}\lambda^\alpha_{12} + \lambda^\alpha_{22}\lambda^\alpha_{11} - \lambda_{12}\lambda_{21}\right]$$

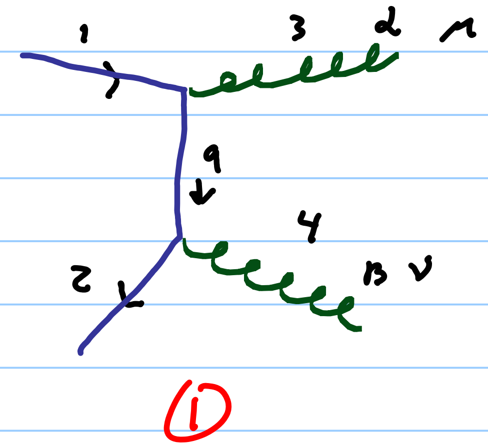
$$= \frac{1}{4}\left\{\lambda^\alpha_{11}\lambda^\alpha_{22} - \lambda^\alpha_{12}\lambda^\alpha_{21}\right\}$$

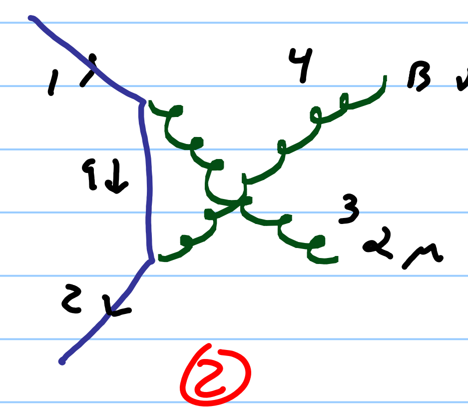
$$= \frac{1}{4}\left\{\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} - \lambda^1_{12}\lambda^1_{21} - \lambda^2_{12}\lambda^2_{21}\right\} = -\frac{2}{3}$$

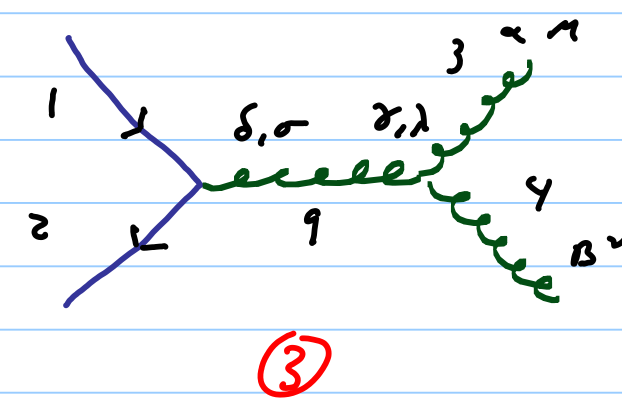
QCD

$$\left.\begin{array}{lll} V_{qq}(r) = -\frac{2}{3}\frac{\alpha_s}{r} & & \text{triplet} \\ V_{qq}(r) \quad \frac{1}{3}\frac{\alpha_s}{r} & & \text{sextet} \end{array}\right] \text{ not seen...}$$

PAIR ANNIHILATION







①: $$-i\mathcal{M}_1 = \bar{v}(2)\, c_2^{\dagger} \left[-i\frac{g_s}{2}\lambda^{\beta}\gamma^{\nu}\right]\left[\varepsilon^{*}_{4\nu}\, a_4^{\beta *}\right]\left[\frac{i(\not{q}+m)}{q^2-m^2}\right]$$
$$\times\left[-i\frac{g_s}{2}\lambda^{\alpha}\gamma^{\mu}\right]\left[\varepsilon_3^{*}\, a_3^{\alpha *}\right] u(1)\, c_1$$

QCD

$q = p_1 - p_3 \quad \rightarrow$

$q^2 - m^2 = p_1^2 - 2\, p_1 \cdot p_3 + p_3^2 - m^2 = -2\, p_1 \cdot p_3$

$$\mathcal{M}_1 = -\frac{g_s^2}{8} \frac{1}{p_1 \cdot p_3} \bar{v}(2) \left[\not{\epsilon}_4 (\not{p}_1 - \not{p}_3 + m) \not{\epsilon}_3 \right] u(1)$$
$$\times\, a_3^\alpha a_4^\beta \left(c_2^T \lambda^\beta \lambda^\alpha c_1 \right)$$

Diagram 2 : switch $3 \longleftrightarrow 4$

colour factor: $a_3^\alpha a_4^\beta \left(c_2^T \lambda^\alpha \lambda^\beta c_1 \right)$

3 : $-i\mathcal{M}_3 = \bar{v}(2)\, c_2^T \left[-i \frac{g_s}{2} \lambda^\delta \gamma_\mu \right] u(1)\, c_1 \left[-i g^{\sigma\lambda} \frac{\delta^{\delta\lambda}}{q^2} \right]$

$\cdot \left[-g_s f^{\alpha\beta\lambda} \left[g_{\mu\nu} (-p_3 + p_4)_\lambda + g_{\nu\lambda} (-p_4 - q)_\mu + g_{\lambda\mu} (q + p_3)_\nu \right] \right]$

$\cdot\ \epsilon_3^\mu a_3^\alpha \ \epsilon_4^\nu a_4^\beta$ 4

QCD

$$q = p_3 + p_4 \quad , \quad q^2 = 2 p_3 \cdot p_4 \quad , \quad \varepsilon_3 \cdot p_3 = 0 = \varepsilon_4 \cdot p_4$$

$$\mathcal{M}_3 = \frac{i g_s^2}{4} \frac{1}{p_3 \cdot p_4} \bar{v}(2) \left[(\varepsilon_3 \cdot \varepsilon_4)(\not{p}_4 - \not{p}_3) + 2 (p_3 \cdot \varepsilon_4) \not{\varepsilon}_3 - 2 (p_4 \cdot \varepsilon_3) \not{\varepsilon}_4 \right] u(1)$$
$$\times f^{\alpha\beta\lambda} a_3^{\alpha} a_4^{\beta} (c_2^{\dagger} \lambda^{\gamma} c_1) \qquad \text{(B)}$$

PROBLEM SET #3, problem 2: show how we get (B) from (A) (Griffiths)

To simplify things, we'll assume that the particles are at rest.

$$p_1 = p_2 = (m, 0) \quad , \quad p_3 = (E, p) \quad , \quad p_4 = (E, -p)$$

$$p_1 \cdot p_3 = p_1 \cdot p_4 = m^2 \quad , \quad p_3 \cdot p_4 = 2m^2$$

$$p_3 \cdot \varepsilon_4 = -\vec{p} \cdot \varepsilon_4 = -p_4 \cdot \varepsilon_4 = 0 \qquad (\text{same for } p_4 \cdot \varepsilon_3)$$

QCD

We now get a Total amplitude :

$$\mathcal{M} = -\frac{g_s}{8m^2} a_3^\alpha a_4^\beta \, \bar{v}(2) c_2^\dagger \left[\not{\varepsilon}_3 \not{\varepsilon}_4 \not{p}_4 \lambda^\alpha \lambda^\beta + \not{\varepsilon}_4 \not{\varepsilon}_3 \not{p}_3 \lambda^\beta \lambda^\alpha \right.$$
$$\left. - i (\varepsilon_3 \cdot \varepsilon_4)(\not{p}_4 - \not{p}_3) f^{\alpha\beta\gamma} \lambda^\gamma \right] c_1 v(1)$$

orient coordinates with $\vec{p}$ along z :

$$\not{p}_3 = m(\gamma^0 - \gamma^3) \;, \quad \not{p}_4 = m(\gamma^0 + \gamma^3) \;, \quad (\not{p}_4 - \not{p}_3) = 2m\gamma^3$$

using $\quad \not{\varepsilon}_3 \not{\varepsilon}_4 = -(\varepsilon_3 \cdot \varepsilon_4) - i(\varepsilon_3 \times \varepsilon_4) \cdot \Sigma$

$\quad \not{\varepsilon}_4 \not{\varepsilon}_3 = -(\varepsilon_3 \cdot \varepsilon_4) + i(\varepsilon_3 \times \varepsilon_4) \cdot \Sigma$

$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$

we get

$$\mathcal{M} = \frac{g_s^2}{8m} a_3^\alpha a_4^\beta \, \bar{v}(2) c_2^\dagger \left\{ (\varepsilon_3 \cdot \varepsilon_4) \{ \lambda^\alpha, \lambda^\beta \} \gamma^0 \right.$$
$$\left. + i (\varepsilon_3 \times \varepsilon_4) \cdot \Sigma \left([\lambda^\alpha, \lambda^\beta] \gamma^0 + \{ \lambda^\alpha, \lambda^\beta \} \gamma^3 \right) \right\} c_1 v(1)$$

$\rightarrow$ same as QED result with $\lambda = 1$, no colour states

QCD

Now we put the quarks in singlet state (spin 0)

$$\mathcal{M} = (\mathcal{M}_{\uparrow\downarrow} - \mathcal{M}_{\downarrow\uparrow}) / \sqrt{2}$$

For $\mathcal{M}_{\uparrow\downarrow}$ we have: $\bar{v}(2)\gamma^0 u(1) = \bar{v}(2)\Sigma\gamma^0 u(1) = 0$

$$\bar{v}(2)\Sigma\gamma^3 u(1) = -2m\hat{z}$$

$$\mathcal{M}_{\downarrow\uparrow} = -\mathcal{M}_{\uparrow\downarrow}$$

$$\mathcal{M} = -i\sqrt{2}\,\frac{g_s^2}{4}\,(\epsilon_3 \times \epsilon_4)_z\, a_3^{\alpha} a_4^{\beta}\,(c_2^{\dagger}\{\lambda^{\alpha},\lambda^{\beta}\}c_1)$$

$$\text{colour Factor} = \frac{1}{8}\, a_3^{\alpha} a_4^{\beta}\,(c_2^{\dagger}\{\lambda^{\alpha},\lambda^{\beta}\}c_1)$$

Problem set 3, problem #3

if quarks occupy colour singlet state $\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})$

Find the colour Factor.

QCD

$$\mathcal{M} = -4 \cdot \sqrt{2/3}\ g_s^2$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \left[\frac{1}{8\pi(E_1+E_2)}\right]^2 \frac{|p_f|}{|p_i|} |\mathcal{M}|^2$$

$$\Rightarrow \sigma \propto \frac{\alpha_s^2}{s} \quad *$$

PS 3, problem 4: 9.14 of Griffiths

→ gg scattering! good Times...

PS 3 due April 15th 1 pm