LECTURE 7: Weak Interactions (Part I)

Overview:

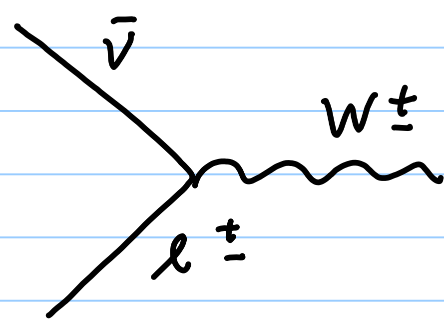
-Inverse muon decay

-Muon Decay

(This lecture mostly follows Griffiths Chapter 10 and Quigg Chapter 6)

Weak Interactions (inverse muon decay)

We will now consider simple processes that involve the weak charged current and leptons



$\rightarrow$ vertex factor $\dfrac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$

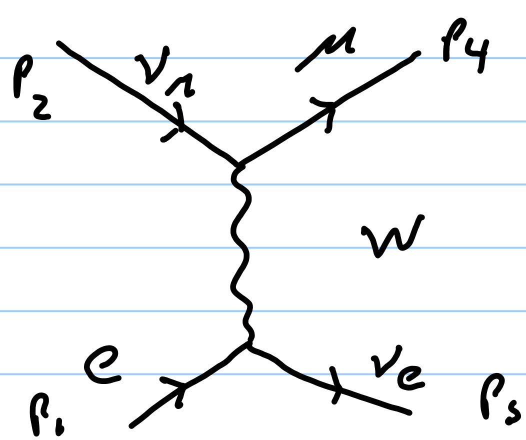
propagator: $\dfrac{-i\left(g_{\mu\nu} - q_\mu q_\nu/M_W^2\right)}{q^2 - M_W^2}$

$\hookrightarrow$ simplifies to $\dfrac{ig_{\mu\nu}}{M_W^2}$

if $q \ll M_W$

Consider the reaction

$\nu_\mu + e^- \rightarrow \mu^- + \nu_e$



$$\mathcal{M} = \frac{g_w^2}{8M_W^2}\left[\bar{u}(3)\gamma^\mu(1-\gamma_5)u(1)\right] \cdot \left[\bar{u}(4)\gamma_\mu(1-\gamma_5)u(2)\right]$$

Weak Interactions (inverse muon decay)

As before we use the relation:

$$\sum_{\text{spins}} [\bar{u}(3)\gamma_\mu(1-\gamma_5)u(1)][\bar{u}(3)\gamma_\nu(1-\gamma_5)u(1)]^*$$
$$= Tr[\gamma_\mu(1-\gamma_5)(\not{p}_1+m_1)\gamma_\nu(1-\gamma_5)(\not{p}_3+m_3)]$$

$$\sum_{\text{spins}} |M|^2 = \left(\frac{g_w^2}{8M_w^2}\right) Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma^5)\not{p}_3] \times$$
$$Tr[\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma_5)(\not{p}_4+m_\mu)]$$

$$\boxed{\begin{array}{l}\text{Trace theorems with } \gamma_5: \\ Tr(\gamma_5)=0 \quad , \quad Tr(\gamma^5 \not{a}\not{b})=0 \\ Tr(\gamma_5\gamma_\mu\gamma_\nu)=0 \quad ; \quad Tr(\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma)=4i\varepsilon_{\mu\nu\lambda\sigma} \\ Tr(\gamma_5\not{a}\not{b}\not{c}\not{d}) = 4i\varepsilon_{\mu\nu\lambda\sigma}a^\mu b^\nu c^\lambda d^\sigma \quad , \quad Tr(\gamma_5\gamma_\mu\gamma_\nu\gamma_\sigma)=0\end{array}}$$

Weak Interactions (inverse muon decay)

Terms from First Trace

1. $\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 = 4(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_1^\nu p_3^\mu)$
2. $\gamma^\mu \not{p}_1 \gamma^\nu \cdot -\gamma^5 \not{p}_3 = 4 i \varepsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma}$
3. $\gamma^\mu(-\gamma^5) \not{p}_1 \gamma^\nu \not{p}_3 =$ (2) $\to$ move γ^5 by 2 positions
4. $\gamma^\mu(-\gamma^5) \not{p}_1 \gamma^\nu (-\gamma^5) \not{p}_3 =$ (1)

$$= \gamma^5 \gamma^\mu \not{p}_1 \gamma^\nu (-\gamma^5) \not{p}_3$$

$$= \gamma^5 \gamma^5 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 = \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3$$

1st Trace $= 8[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1 \cdot p_3) - i\varepsilon^{\mu\nu\lambda\sigma} p_{1\lambda} p_{3\sigma}]$

2nd Trace $= 8[p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu}(p_2 \cdot p_4) - i\varepsilon_{\mu\nu\kappa\tau} p_2^\kappa p_4^\tau]$

note that: $\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\kappa\tau} = -2(\delta^\lambda_\kappa \delta^\sigma_\tau - \delta^\lambda_\tau \delta^\sigma_\kappa)$

Weak Interactions (inverse muon decay)

Multiplying the two trace results we get:

$$= \Big[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$+ (p_1 \cdot p_4)(p_3 \cdot p_2) + (p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$- (p_2 \cdot p_4)(p_1 \cdot p_3) - (p_2 \cdot p_4)(p_1 \cdot p_3) + 4(p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$+ 2(p_1 \cdot p_2)(p_3 p_4) - 2(p_1 p_4)(p_3 p_2)$$

$$\Rightarrow \sum_{spins} |M|^2 = 4\left(\frac{g_w}{M_w}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

in CM frame, neglecting electron mass and muon mass

$$p_1 = (p_i, 0, 0, \vec{p_i}), \quad p_2 = (p_i, 0, 0, -\vec{p_i})$$

$$p_3 = (p_F, 0, 0, \vec{p_F}), \quad p_4 = (p_F, 0, 0, -p_F)$$

$$p_1 \cdot p_2 = 2p_i^2, \quad p_3 \cdot p_4 = 2p_F^2$$

Weak Interactions (inverse muon decay)

$$2p_i = 2p_f = 2E$$

$$\Rightarrow \sum_{\text{spins}} |M|^2 = 4 \cdot 4E^4 \left(\frac{g_w}{M_w}\right)^4 \quad \rightarrow \quad 4E^2 = s$$

$$G_F = \frac{\sqrt{2}}{8} \frac{g_w^2}{M_w^2} \quad , \quad G_F^2 = \frac{1}{32} \frac{g_w^4}{M_w^4}$$

$$\sum_{\text{spins}} |M|^2 = 32 G_F^2 \cdot s^2$$

$\sum_{\text{spins}}$ → neutrino one spin state

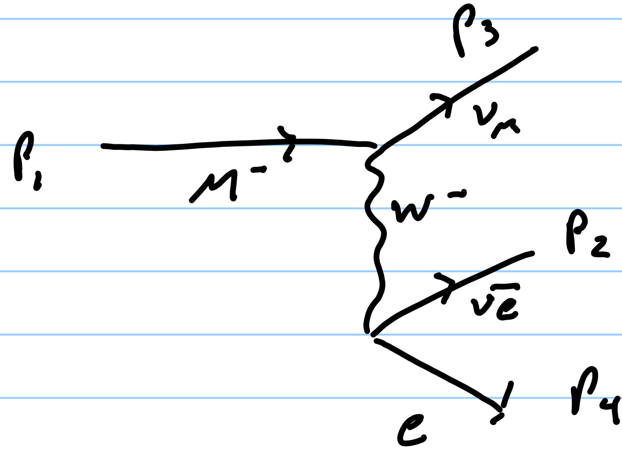
→ electron two spin states

average over initial spins adds $\frac{1}{2}$ factor

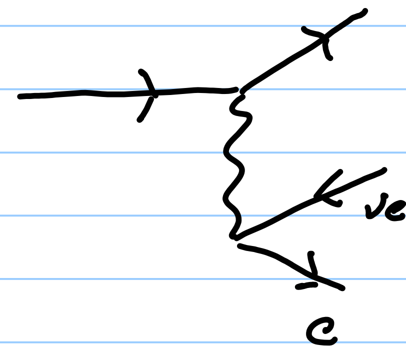
$$\langle |M|^2 \rangle = 16\, G_F^2 s^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \overline{|M|^2} = \frac{1}{4\pi^2} G_F^2 s \quad , \quad \sigma = \frac{G_F^2 s}{\pi}$$

Muon decay



or



$$\mathcal{M} = \frac{g_w^2}{8M_W^2}\left[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)\right]\left[\bar{u}(4)\gamma_\mu(1-\gamma^5)v(2)\right]$$

this is the same $\mathcal{M}$ as we got for inverse muon decay.

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_W}\right)^4 (p_1\cdot p_2)(p_3\cdot p_4)$$

$p_1 = (m_\mu, 0, 0, 0) \quad \Rightarrow \quad p_1\cdot p_2 = m_\mu E_2$

Muon decay (cont)

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2 p_3 \cdot p_4 = m_e^2 + 2 p_3 \cdot p_4$$

$$= (p_1 - p_2)^2 = m_\mu^2 - 2 p_1 \cdot p_2$$

$$\Rightarrow \quad p_3 \cdot p_4 = \left(\frac{m_\mu^2 - m_e^2}{2} \right) - m_\mu E_2$$

$m_e \sim 200$ less than m_μ $\rightarrow$ we neglect ...

$$\langle |M|^2 \rangle = \left(\frac{g_w}{m_w} \right)^2 m_\mu^2 E_2 \left(m_\mu - 2E_2 \right)$$

$$d\Gamma = \frac{\langle |M|^2 \rangle}{2 m_\mu} \left(\frac{d^3 \vec{p}_2}{(2\pi)^3 2E_2} \right) \left(\frac{d^3 \vec{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{d^3 \vec{p}_4}{(2\pi)^3 2E_4} \right) (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 - p_4)$$

$$E_2 = p_2 \quad , \quad E_3 = p_3 \quad , \quad E_4 = p_4$$

$$\delta^4 (\quad) = \delta (m_\mu - E_2 - E_3 - E_4) \, \delta^3 (p_2 + p_3 + p_4)$$

integrate over p_3:

Muon decay (cont.)

$$d\Gamma = \frac{\langle|M|^2\rangle}{16(2\pi)^5 m_\mu} \frac{d^3\vec{p}_2\, d^3\vec{p}_4}{E_2 E_3 E_4}\, \delta(m_\mu - E_2 - E_3 - E_4)$$

E_3 now is $= |p_2 + p_4|$

$$|p_2+p_4|^2 = E_3^2 = p_2^2 + p_4^2 + 2\, p_2 \cdot p_4$$

$$= (E_2^2 + E_4^2 + 2E_2E_4\cos\theta)$$ $\rightarrow$ set polar axis along $\vec{p}_4$

$$d^3\vec{p}_2 = E_2^2\, dE_2 \sin\theta\, d\theta\, d\varphi$$

φ integral $= 2\pi$

θ integral we set $x = \sqrt{E_2^2 + E_4^2 + 2E_2E_4\cos\theta} = E_3$

$$dx = \frac{-E_2E_4\sin\theta\, d\theta}{E_3}$$

$$\int_0^\pi \frac{\sin\theta}{E_3} d\theta\, \delta(m_\mu - E_2 - E_3 - E_4)$$

becomes: $\displaystyle \frac{1}{E_2E_4}\int_{x_-}^{x_+} \delta(m_\mu - x - E_2 - E_4)\, dx$

Muon decay (cont.)

$= \frac{1}{E_2 E_4}$ if $x_- < \delta(m_\mu - E_2 - E_4) < x_+$

$= 0$ otherwise

now $x_\pm = \sqrt{E_2^2 + E_4^2 \pm 2E_2E_4} = |E_2 \pm E_4|$

so: $|E_2 - E_4| < (m_\mu - E_2 - E_4) < E_2 + E_4$

$= \frac{1}{2}\left[|E_2 - E_4| + E_2 + E_4 \right] < \frac{m_\mu}{2} < E_2 + E_4$

$\longrightarrow$ $E_2 < \frac{m_\mu}{2}$, $(E_2 + E_4) > \frac{m_\mu}{2}$

$E_4 < \frac{m_\mu}{2}$

Muon Decay (cont.)

Putting everything back together we get:

$$d\Gamma = \frac{\langle |M|^2 \rangle}{(4\pi)^4 m_\mu} \, dE_2 \, \frac{d^3 p_4}{E_4^2}$$

$$\langle |M|^2 \rangle = \left(\frac{g_w}{M_w}\right)^4 m_\mu^2 E_2 (m_\mu - 2E_2)$$

$$d\Gamma = \frac{m_\mu}{(4\pi)^4} \left(\frac{g_w}{M_w}\right)^4 \frac{dE_2}{E_4^2} E_2 (m_\mu - 2E_2) \, d^3 p_4$$

integrate over E_2

$$\left(\frac{g_w}{4\pi M_w}\right)^4 m_\mu \frac{d^3 p_4}{E_4^2} \int_{1/2 m_\mu - E_4}^{1/2 m_\mu} E_2 (m_\mu - 2E_2) \, dE_2$$

$$= \left(\frac{g_w}{4\pi M_w}\right)^4 m_\mu \left(\frac{m_\mu}{2} - \frac{2}{3} E_4\right) d^3 p_4$$

Muon decay

using : $d^3 p_4 = 4\pi E_4^2 \, dE_4$

we get $\Gamma = \left(\frac{g_w}{m_w}\right)^4 \frac{m_\mu^2 E_4^2}{2(4\pi)^3} \left(1 - \frac{4E_4}{3m_\mu}\right) dE_4$

$$\Gamma = \left(\frac{g_w}{m_w}\right)^4 \frac{m_\mu^2}{2(4\pi)^3} \int_0^{1/2 m_\mu} \left(1 - \frac{4E_4}{3m_\mu}\right) E_4^2 \, dE_4$$

int : $\frac{E_4^3}{3} \to \frac{m_\mu^3}{24}$, $\frac{-4E_4^4}{3 \cdot 4 m_\mu} = \frac{-4 m_\mu^3}{4 \cdot 3 \cdot 16} = \frac{m_\mu^3}{48}$

$\frac{m_\mu^3}{48}$

$$\Rightarrow \Gamma = \left(\frac{g_w m_\mu}{m_w}\right)^4 \frac{m_\mu}{96(4\pi)^3} , \quad \tau = \frac{1}{\Gamma}$$

$$G_F = \frac{\sqrt{2}}{8}\left(\frac{g_w}{m_w}\right)^2 \Rightarrow \tau = \frac{192\pi^3}{G^2 m_\mu^5} = \left(\frac{192\pi^3 \hbar^7}{G^2 m_\mu^5 c^4}\right)$$

$$\boxed{\simeq 2 \, \mu s}$$