

## LECTURE 18: QCD (Part 1)

### Overview:

- QCD introduction
- Feynman rules
- Calculating colour factors
- quark anti-quark annihilation

(I used Quigg and mostly Griffiths as references)

# QCD (I)

Evidence that quarks come in 3 different "colours" include

- spin-stat. problem for baryons
- cross section  $e^+e^- \rightarrow$  hadrons
- $r$  branching ratios
- $\pi^0$  lifetime
- anomaly cancellation

Could we use  $U(3)$ ,  $SO(3)$  as candidate gauge groups for the strong interaction?

- $SO(3) \rightarrow$  no asymptotic freedom
- $\rightarrow$  existence of diquark states, no distinction between quarks and anti-quarks

## QCD (II)

$U(3)$  we get a singlet gauge boson  $\rightarrow$  long-range interaction.

For the Lagrangian:

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m)\psi - \frac{1}{2} \text{Tr} (G_{\mu\nu} G^{\mu\nu})$$

$$\psi = \begin{pmatrix} \psi \\ \psi \\ \psi \end{pmatrix}$$

$$D_\mu = \partial_\mu + ig B_\mu$$

$\rightarrow$   $3 \times 3$  matrix:  $B_\mu = \frac{1}{2} \lambda \cdot b_\mu = \frac{1}{2} \lambda^a b_\mu^a$

$b_\mu$ : 8 colour gauge fields

$$G_{\mu\nu} = \frac{1}{2} G_{\mu\nu} \cdot \lambda = \frac{1}{2} G_{\mu\nu}^a \lambda^a$$

$$= (ig)^{-1} [D_\nu, D_\mu] = \partial_\nu B_\mu - \partial_\mu B_\nu + ig [B_\nu, B_\mu]$$

## QCD (III)

Properties of the  $\lambda$  matrices:

$$\begin{aligned} \text{Tr}(\lambda^a \lambda^a) &= 0 & F_{123} &= 1, & F_{458} &= F_{678} = \sqrt{3}/2 \\ \text{Tr}(\lambda^k \lambda^l) &= 2\delta^{kl} & F_{177} &= F_{276} = F_{345} = F_{516} = F_{637} = 1/2 \\ [\lambda^i, \lambda^k] &= 2i f^{ijk} \lambda^k \end{aligned}$$

$$G_{\mu\nu}^a = \partial_\nu b_\mu^a - \partial_\mu b_\nu^a + g f^{ijk} b_\mu^j b_\nu^k$$

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

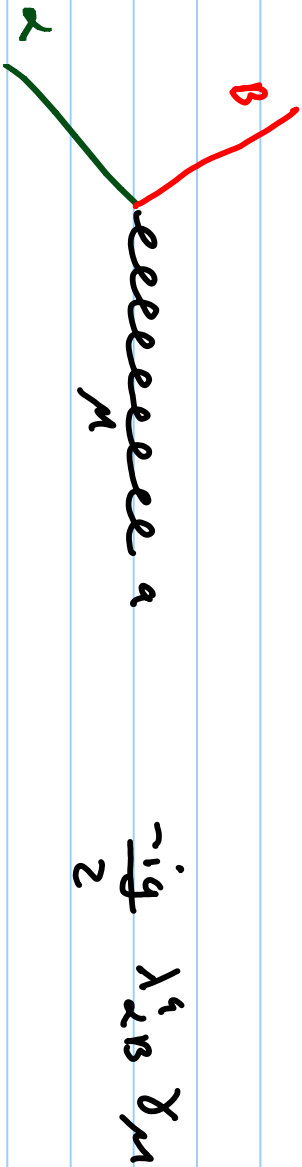
$$\begin{aligned} \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

# QCD IV

(5)

quark - gluon interaction:  $\mathcal{L}_{int} = -\frac{g}{2} b^a \bar{\psi} \gamma^m \lambda^a \psi$



→ one gluon exchange force prop.  $\frac{g^2}{4} \sum_a \lambda_{ab}^a \lambda_{cd}^a$

$$\alpha + \gamma \rightarrow \beta + \delta$$

Gluons are massless particles with spin 1 which we will represent by pol. vector  $\epsilon^m$ .

→  $\epsilon^m p_m = 0$  (Lorentz Condition)

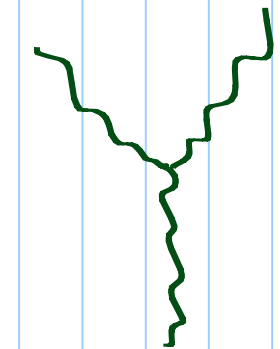
# QCD (VI)

We will also use Coulomb gauge:  $\vec{\epsilon}' = 0$   $\vec{\epsilon} \cdot \vec{p} = 0$

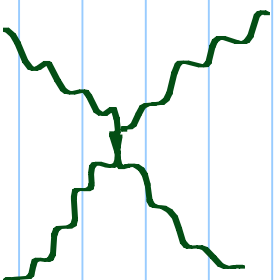
To describe colour state of gluon, we'll use colour vectors:

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \equiv |1\rangle \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \equiv |4\rangle$$

Non-Abelian group  $\rightarrow$  gauge mediators interact with themselves:



3-gluon vertex



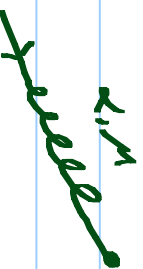
4-gluon vertex

# QCD (VII)

⑦

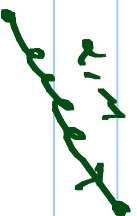
## Feynman Rules

incoming gluon:



$$\epsilon_{\mu}(p) e^{\alpha}$$

outgoing gluon:



$$\epsilon_{\mu}^*(p) e^{\alpha*}$$

## Propagators

$q\bar{q}$



$$\frac{i(\not{q} + m)}{q^2 - m^2}$$

gluon

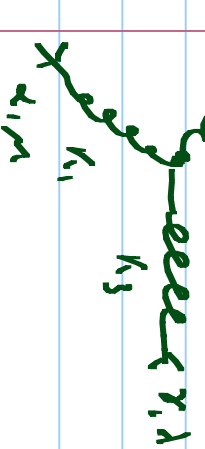


$$\frac{-i g_{\mu\nu} \delta_{ab}}{q^2}$$

## Vertices



$$-i g_s \lambda^a \gamma_{\mu}$$

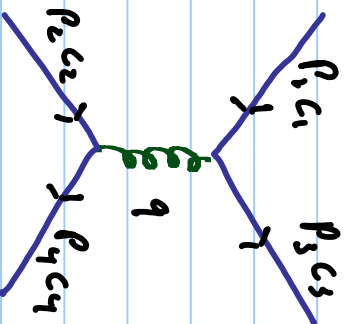


$$-g_s f^{abc} \gamma_{\mu} [g_{\mu\nu} (K_1 - K_2)_{\nu} + g_{\nu\lambda} (K_2 - K_3)_{\lambda} + g_{\lambda\mu} (K_3 - K_1)_{\lambda}]$$

# QCD

(8)

Consider  $u + \bar{d} \rightarrow u + \bar{d}$



$$-iM = [\bar{u}(3) c_3^T] \left[ -i \frac{g_s}{2} \lambda^A \gamma^A \right] [u(1) c_1] \left[ -i \frac{g_s \gamma^B}{q^2} \delta^{AB} \right]$$

$$\times [\bar{v}(2) c_2^T] \left[ -i \frac{g_s}{2} \lambda^B \gamma^B \right] [v(4) c_4]$$

$$M = -\frac{g_s^2}{4} \frac{1}{q^2} [\bar{u}(3) \gamma^A u(1)] [\bar{v}(2) \gamma^B v(4)] \underbrace{(c_3^T \lambda^A c_1) (c_2^T \lambda^B c_4)}_{\text{colour Factor}}$$

→ same as electron-positron scattering with  $g_e \rightarrow g_s$   
 but with extra colour factor

Potential similar to EM case  $V \propto -\frac{F_i^j}{r}$



# QCD

(7)

— we'll consider the colour octet case e.g.  $\bar{r}\bar{b}$

$$c_1 = c_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = c_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$F = \frac{1}{4} \begin{bmatrix} 11601\lambda^2 & (1) \\ & (0) \end{bmatrix} \begin{bmatrix} 60161\lambda^2 & (0) \\ & (0) \end{bmatrix} = \frac{1}{4} \lambda_{11}^2 \lambda_{22}^2$$

→  $\lambda^3$  and  $\lambda^6$  are the only relevant matrices

$$F = \frac{1}{4} (\lambda_{11}^3 \lambda_{22}^3 + \lambda_{11}^6 \lambda_{11}^6) = -1/6$$

— now we consider the singlet state:  $\frac{1}{\sqrt{3}} (r\bar{r} + b\bar{b} + g\bar{g})$

$$F = \frac{1}{4} \cdot \frac{1}{\sqrt{3}} \left[ \left( c_3^T \lambda^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \begin{bmatrix} 11601\lambda^2 & c_4 \end{bmatrix} + \left( c_3^T \lambda^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \begin{bmatrix} 6101\lambda^2 & c_4 \end{bmatrix} \right. \\ \left. + \left( c_3^T \lambda^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \begin{bmatrix} 10,0,11\lambda^2 & c_4 \end{bmatrix} \right]$$



# QCD

(11)

quark - quark interaction

$$M = -\frac{g_s^2}{4} \frac{1}{f^2} [\bar{u}(1) \gamma^\mu u(1)] [\bar{u}(2) \gamma^\mu u(2)] (c_1^\dagger \lambda^a c_1) (c_2^\dagger \lambda^a c_2)$$

$$F = \frac{1}{4} (c_1^\dagger \lambda^a c_1) (c_2^\dagger \lambda^a c_2)$$

we get a triplet:

$$\left. \begin{array}{l} (rb - br) / \sqrt{2} \\ (bs - sb) / \sqrt{2} \\ (gr - rg) / \sqrt{2} \end{array} \right\} \text{anti-sym combinations}$$

and a sextet:

$$r, g, b, b$$

+ symmetric combinations

QCD

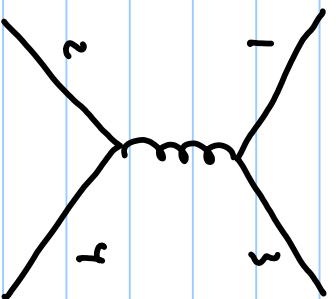
(12)

setlet :  $C_1 = C_2 = C_3 = C_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (for  $r$ )

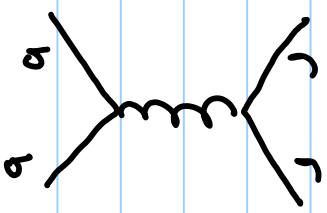
$$F = \frac{1}{4} \left[ C_{1001} \lambda^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \left[ C_{1001} \lambda^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \frac{1}{4} \lambda_{11}^x \lambda_{11}^x$$

$$= \frac{1}{4} \left[ \lambda_{11}^3 \lambda_{11}^3 + \lambda_{11}^8 \lambda_{11}^8 \right] = \frac{1}{4} \left[ 1 + \frac{1}{3} \right] = \frac{1}{3}$$

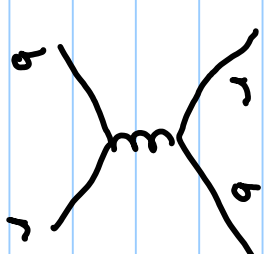
For triplet: For example  $(rb-br)/\sqrt{2}$



:



-



-



QCD

(13)

We get:

$$\begin{aligned} F &= \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \left[ \left[ (1100) \lambda^x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] \left( (0101) \lambda^x \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) \right. \\ &\quad - \left[ (0101) \lambda^x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] \left[ (1100) \lambda^x \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &\quad + \left[ (0101) \lambda^x \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[ (1001) \lambda^x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] \\ &\quad - \left[ (1001) \lambda^x \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[ (0101) \lambda^x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] \left. \right] \\ &= \frac{1}{8} \left[ \lambda_{11}^x \lambda_{22}^x - \lambda_{21}^x \lambda_{12}^x + \lambda_{22}^x \lambda_{11}^x - \lambda_{12}^x \lambda_{21}^x \right] \\ &= \frac{1}{4} \left[ \lambda_{11}^x \lambda_{22}^x - \lambda_{12}^x \lambda_{21}^x \right] \\ &= \frac{1}{4} \left[ \lambda_{11}^3 \lambda_{22}^3 + \lambda_{11}^6 \lambda_{22}^6 - \lambda_{12}^1 \lambda_{21}^1 - \lambda_{12}^2 \lambda_{21}^2 \right] = -\frac{2}{3} \end{aligned}$$

QCD

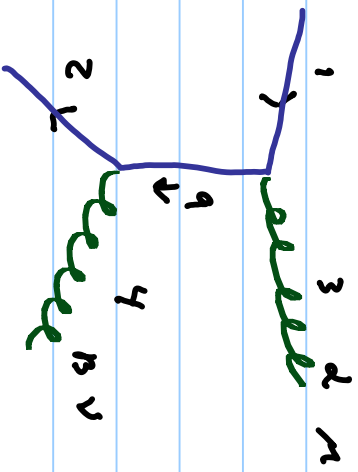
$$V_{qq}(r) = -\frac{2}{3} \frac{d_s}{r} \quad \text{Triplet}$$

$$V_{qq}(r) = \frac{1}{3} \frac{d_s}{r} \quad \text{sextet}$$

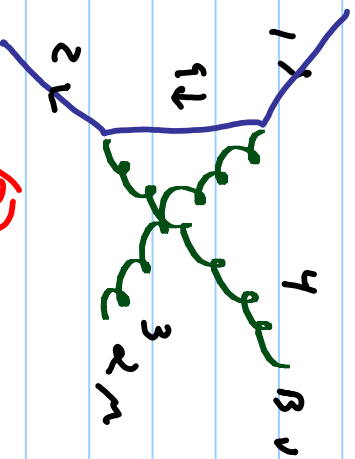
not seen ...

(14)

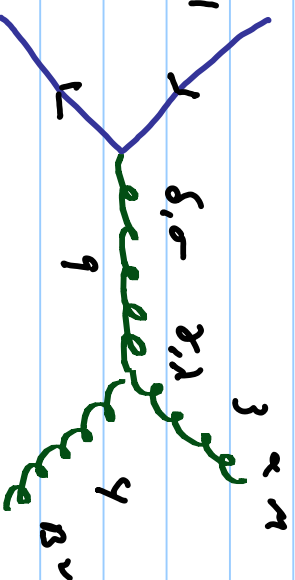
### Pair ANNIHILATION



(1)



(2)



(3)

$$\textcircled{1} : -iM_1 = \bar{v}(2) c_2^t \left[ -i \frac{g_s}{2} \lambda^B \gamma^\nu \right] \left[ \epsilon_{\nu\mu}^a a_{\mu}^{B*} \right] \left[ \frac{i(g + m)}{q^2 - m^2} \right] \times \left[ -i \frac{g_s}{2} \lambda^A \gamma^\mu \right] \left[ \epsilon_{\mu\sigma}^a a_{\sigma}^{A*} \right] v(1) c_1$$

QCD

(13)

$$q = p_1 - p_3 \rightarrow$$

$$q^2 - m^2 = p_1^2 - 2 p_1 \cdot p_3 + p_3^2 - m^2 = -2 p_1 \cdot p_3$$

$$M_1 = -\frac{g_s^2}{8} \frac{1}{p_1 \cdot p_3} \bar{v}(2) [ \cancel{\not{\epsilon}_4} (p_1 - p_3 + m) \cancel{\not{\epsilon}_3} ] v(1) \\ \times a_3^\alpha a_4^\beta (c_2^\dagger \lambda^\alpha \lambda^\beta c_1)$$

Diagram 2: switch  $3 \leftrightarrow 4$

colour factor:  $a_3^\alpha a_4^\beta (c_2^\dagger \lambda^\alpha \lambda^\beta c_1)$

$$(3): -i M_3 = \bar{v}(2) c_2^\dagger \left[ -i \frac{g_s}{2} \lambda^\alpha \delta_{\alpha\beta} \right] v(1) c_1 \left[ -i g \frac{\sigma^{\beta\gamma}}{q^2} \right]$$

$$\cdot \left[ -g_s F^{\alpha\beta\gamma} \left[ g_{\mu\nu} (-p_3 + p_4)_\nu + g_{\nu\lambda} (-p_4 - q)_\lambda + g_{\lambda\mu} (q + p_3)_\mu \right] \right] \\ \cdot \epsilon_3^\mu \epsilon_3^\alpha \epsilon_4^\nu \epsilon_4^\beta \quad (A)$$

QCD

(16)

$$q^1 = p_3 + p_4, \quad q^2 = 2p_3 \cdot p_4, \quad \epsilon_3 \cdot p_3 = 0 = \epsilon_4 \cdot p_4$$

$$M_3 = i \frac{g_s^2}{4} \frac{1}{p_3 \cdot p_4} \bar{v}(l_2) [(\epsilon_3 \cdot \epsilon_4) (\not{p}_4 - \not{p}_3) + 2(p_3 \cdot \epsilon_4) \epsilon_3 - 2(p_4 \cdot \epsilon_3) \epsilon_4] u(l_1) \\ \times \text{Feynman rules } a_3^\dagger a_4^b (c_2^\dagger \lambda^c c_1) \quad (13)$$

PROBLEM SET #3, problem 2: show how we get (13) from (A) (Griffiths)

To simplify things, we'll assume that the particles are at rest.

$$p_1 = p_2 = (m, 0), \quad p_3 = (E, p), \quad p_4 = (E, -p)$$

$$p_1 \cdot p_3 = p_1 \cdot p_4 = m^2, \quad p_3 \cdot p_4 = 2m^2$$

$$p_3 \cdot \epsilon_4 = -\vec{p} \cdot \epsilon_4 = -p_4 \cdot \epsilon_4 = 0 \quad (\text{same for } p_4 \cdot \epsilon_3)$$



QCD

(17)

We now get a Total amplitude :

$$M = \frac{-g_s^2}{8m^2} a_3^2 a_4^2 \bar{v}(2) c_2^+ \left[ \not{x}_3 \not{x}_4 \not{p}_4 \lambda^2 \lambda^0 + \not{x}_4 \not{x}_3 \not{p}_3 \lambda^0 \lambda^2 \right. \\ \left. - i (\not{x}_3 \cdot \not{x}_4) (\not{p}_4 - \not{p}_3) \not{x}^{\alpha\beta} \lambda^{\alpha\beta} \right] c_1 v(1)$$

orient coordinates with  $\vec{p}$  along z :

$$p_3 = m(\gamma^0 - \gamma^3), \quad p_4 = m(\gamma^0 + \gamma^3), \quad (p_4 - p_3) = 2m\gamma^3$$

using

$$\not{x}_3 \not{x}_4 = -(\not{x}_3 \cdot \not{x}_4) - i(\not{x}_3 \times \not{x}_4) \cdot \Sigma$$

$$\not{x}_4 \not{x}_3 = -(\not{x}_3 \cdot \not{x}_4) + i(\not{x}_3 \times \not{x}_4) \cdot \Sigma$$

$$\Sigma = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}$$

we get 
$$M = \frac{g_s^2}{8m} a_3^2 a_4^2 \bar{v}(2) c_2^+ \left[ (\not{x}_3 \cdot \not{x}_4) \{ \lambda^2, \lambda^0 \} \gamma^0 \right.$$

$$\left. + i(\not{x}_3 \times \not{x}_4) \cdot \Sigma \left[ \lambda^2, \lambda^0 \right] \gamma^0 + \{ \lambda^2, \lambda^0 \} \gamma^3 \right] c_1 v(1)$$

→ same as QED result with  $\lambda=1$ , no colour states

## QCD

Now we put the quarks in singlet state (spin 0)

$$M = (M_{\uparrow\downarrow} - M_{\downarrow\uparrow}) / \sqrt{2}$$

For  $M_{\uparrow\downarrow}$  we have:  $\bar{v}(2) \gamma^0 v(1) = \bar{v}(2) \sum \gamma^0 v(1) = 0$

$$\bar{v}(2) \sum \gamma^3 v(1) = -2m \hat{z}$$

$$M_{\downarrow\uparrow} = -M_{\uparrow\downarrow}$$

$$M = -i\sqrt{2} \frac{g_s^2}{4} (\epsilon_3 \times \epsilon_4)_z a_3^2 a_4^B (c_2^t \{ \lambda^4, \lambda^0 \} c_1)$$

$$\text{colour factor} = \frac{1}{8} a_3^2 a_4^B (c_2^t \{ \lambda^4, \lambda^0 \} c_1)$$

Problem set 3, problem #3

if quarks occupy colour singlet state  $\frac{1}{\sqrt{3}} (r\bar{r} + b\bar{b} + g\bar{g})$

Find the colour factor.

QCD

(19)

$$M = -4 \cdot \sqrt{2/3} g_s^2$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \left[ \frac{1}{8\pi(E_1 + E_2)} \right]^2 \frac{|p_{e1}|}{|p_{i1}|} |M|^2$$

$$\Rightarrow \sigma \propto \frac{2^2}{5} \neq$$

PROBLEM SET #3

PROBLEM 1: THOMSON 10.6

PROBLEM 2: SEE PAGE 16

PROBLEM 3: SEE PAGE 18

→ Due Dec 19<sup>th</sup> 2014

