REVIEW Part 3: Calculation of QED Cross Sections and Decay Rates

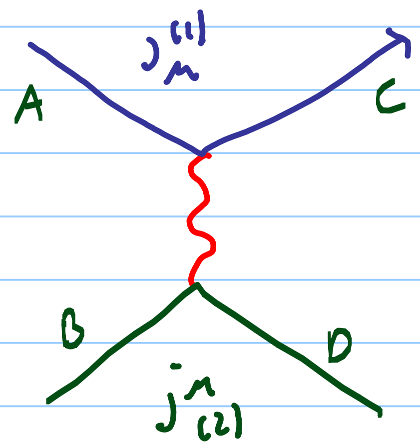
Overview:

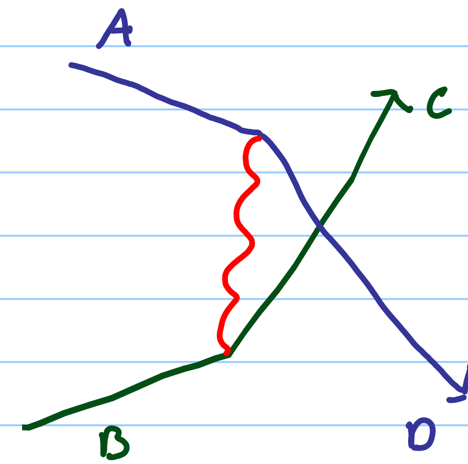
- Cross section calculation for electron-electron scattering
- Cross section calculation for electron-muon scattering
- Cross section calculation for Compton scattering and electron-positron annihilation

(This lecture mostly follows Halzen and Martin Chap. 6 and Griffiths Chap. 6-7)

Cross Section Calculations (adding spin)

$e^-e^- \to e^-e^-$ scattering





$$T_{fi} = -i \int j^{(1)}_\mu(x) \left(-\frac{1}{q^2}\right) j^\mu_{(2)}(x)\, d^4x \quad , \quad q = (p_A - p_C)$$

$$= -i\left(-e\bar{u}_C \gamma_\mu u_A\right)\left(-\frac{1}{q^2}\right)\left(-e\bar{u}_D \gamma^\mu u_B\right)(2\pi)^4 \delta^{(4)}(p_A + p_B - p_C - p_D)$$

$$T_{fi} = -i(2\pi)^4 \delta^{(4)}(p_A + p_B - p_C - p_D)\, \mathcal{M}$$

ADD SECOND DIAGRAM ($C \leftrightarrow D$ and minus sign because we swap identical fermions)

Cross Section Calculations (adding spin)

We get for $\mathcal{M}_{e^-e^-}$:

$$-e^2 \frac{(\bar{u}_C \gamma^\mu u_A)(\bar{u}_D \gamma_\mu u_B)}{(p_A - p_C)^2} + e^2 \frac{(\bar{u}_D \gamma^\mu u_A)(\bar{u}_C \gamma_\mu u_B)}{(p_A - p_D)^2}$$

Unpolarized cross section : average over spins

$$\frac{1}{(2s_A+1)(2s_B+1)} \sum_{\text{spins}} |\mathcal{M}|^2$$

— we take non-rel. limit $p = 0$

$$u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \frac{\sigma \cdot p}{E+m} \chi^{(s)} \end{pmatrix} \qquad E > 0 \;,\; s = 1,2$$

$$N = \sqrt{E+m}$$

$$u^{(s)} = \sqrt{m} \begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix} \qquad \bar{u}^{(s)} = \sqrt{m}\,(\chi^{(s)\dagger}, 0)$$

Cross Section Calculations (adding spin)

with $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$

$$\bar{u}^{(s)} \gamma^0 u^{(s)} = (\sqrt{m}\, \chi^{(s)}, 0) \begin{pmatrix} \sqrt{m}\, \chi^{(s)} \\ 0 \end{pmatrix} = m + m$$

$$\bar{u}^{(s)} \gamma^k u^{(s)} = (0, \sqrt{m}\, \chi^{T(s)}) \begin{pmatrix} \chi^{(s)+} \\ 0 \end{pmatrix} = 0$$

$$\bar{u}^{(s)} \gamma^\mu u^{(s')} = 0 \quad \text{if} \quad s \neq s'$$

$\Rightarrow$ no spin flip! (non-rel.)

$$\Rightarrow \mathcal{M}(\uparrow\uparrow \to \uparrow\uparrow) = \mathcal{M}(\downarrow\downarrow \to \downarrow\downarrow) = -e^2 4m^2 \left(\frac{1}{T} - \frac{1}{U} \right)$$

$$\mathcal{M}(\uparrow\downarrow \to \uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow \to \downarrow\uparrow) = -e^2 4m^2 \frac{1}{T}$$

$$\mathcal{M}(\uparrow\downarrow \to \downarrow\uparrow) = \mathcal{M}(\downarrow\uparrow \to \uparrow\downarrow) = e^2 4m^2 \frac{1}{U}$$

$$|\mathcal{M}|^2 = \frac{1}{4} (4m^2 e^2)^2 \cdot 2 \left[\left(\frac{1}{T} - \frac{1}{U} \right)^2 + \frac{1}{T^2} + \frac{1}{U^2} \right]$$

Cross Section Calculations (adding spin)

From page 12

Review part II

$$T = -2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2}$$

$$u = -2p^2(1+\cos\theta) = -4p^2\cos^2\frac{\theta}{2}$$

$$s = 4(p^2+m^2) \simeq 4m^2$$

centre of mass

From page 6

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{p_F}{p_i}\frac{|M|^2}{s}$$

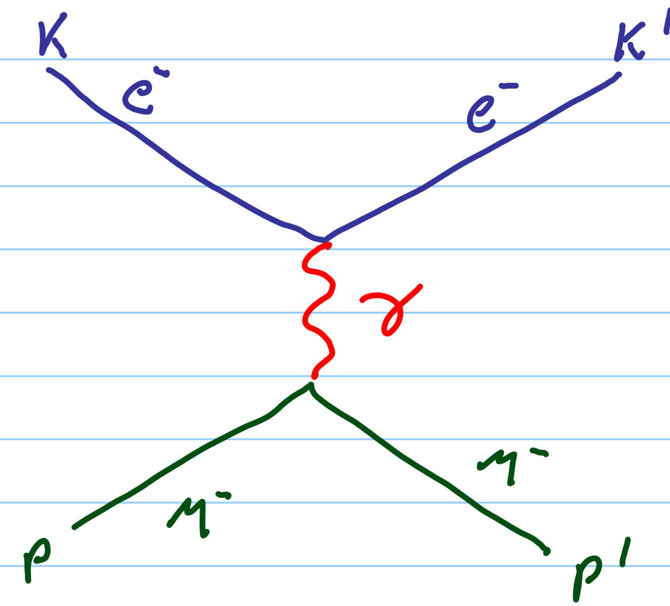
From last page: $|M|^2 = \frac{1}{4}(4m^2e^2)^2 \cdot 2\left[\left(\frac{1}{T}-\frac{1}{u}\right)^2 + \frac{1}{T^2} + \frac{1}{u^2}\right]$

we get:

$$\frac{d\sigma}{d\Omega}_{cm} = \frac{m^2\alpha^2}{16p^4}\left(\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} - \frac{1}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right)$$

Cross Section Calculations (adding spin)

Back to $e\mu$ scattering:



$$\mathcal{M} = -e^2\, \bar{u}(k')\gamma^\mu u(k) \frac{1}{q^2}\, \bar{u}(p')\gamma_\mu u(p)$$

We will write $|\mathcal{M}|^2$ as:

$$|\mathcal{M}|^2 \propto L_e^{\mu\nu} L^{\text{muon}}_{\mu\nu}$$

electron Tensor: $\frac{1}{2}\sum_{\text{spins}} [\bar{u}(k')\gamma^\mu u(k)][\bar{u}(k')\gamma^\nu u(k)]^*$

1×1 matrix so complex conj. = hermitian conj.

$$= [u^T(k')\gamma^0\gamma^\nu u(k)]^\dagger = [u^\dagger(k)\,\gamma^{\nu\dagger}\gamma^0 u(k')]$$

$$= [\bar{u}(k)\gamma^\nu u(k')]$$

we've reversed order of matrix product

Cross Section Calculations (adding spin)

We get

$$L_e^{\mu\nu} = \frac{1}{2} \sum_{s'} \bar{u}_\alpha^{(s')}(k')\,\gamma^\mu_{\alpha\beta} \sum_s \underbrace{u_\beta^{(s)}(k)\,\bar{u}_\gamma^{(s)}(k)}_{(\not{k}+m)_{\beta\gamma}}\,\gamma^\nu_{\gamma\delta}\,u_\delta^{(s')}(k')$$

$$\underbrace{\bar{u}_\alpha^{(s')}(k') \ldots u_\delta^{(s')}(k')}_{(\not{k}'+m)_{\delta\alpha}}$$

$$L_e^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\left((\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^\nu\right)$$

TRACE THEOREMS: $\mathrm{Tr}\,1 = 4$, $\mathrm{Tr}(\not{a}\not{b}) = 4\,a\cdot b$

Trace of odd # of γs $= 0$, $\mathrm{Tr}\,\gamma_5 = 0$, $\mathrm{Tr}(\gamma_5\not{a}\not{b}) = 0$

$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = g^{\mu\nu}$, $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$

Cross Section Calculations (adding spin)

Other relations: $\gamma_\mu \gamma^\mu = 4$, $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$ $\qquad \gamma_\mu \not{a} \not{b} \gamma^\mu = 4 a \cdot b$

$\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2 \not{c} \not{b} \not{a}$

$e\mu$ scattering continued:

$$L_e^{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left((\not{k}' + m) \gamma^\mu (\not{k} + m) \gamma^\nu \right)$$

$$= \frac{1}{2} \mathrm{Tr}(\not{k}' \gamma^\mu \not{k} \gamma^\nu) + \frac{1}{2} m^2 \mathrm{Tr}(\gamma^\mu \gamma^\nu)$$

$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

$$= (k')_\lambda (k)_\sigma \mathrm{Tr}(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma)$$

$$= (k')_\lambda (k)_\sigma 4 (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu})$$

$$= 4 (k'^\mu k^\nu - g^{\mu\nu} (k \cdot k') + k^\mu k'^\nu)$$

Cross Section Calculations (adding spin)

$$L_e^{\mu\nu} = 2\left(k'^{\mu}k^{\nu} + k'^{\nu}k^{\mu} - (k'\cdot k - m^2)\, g^{\mu\nu}\right)$$

$$L^{muon}_{\mu\nu} = 2\left(p'_{\mu}p_{\nu} + p'_{\nu}p_{\mu} - (p'\cdot p - M^2)\, g_{\mu\nu}\right)$$

$$\rightarrow |M|^2 = \frac{8e^4}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') + \ldots\right]$$

$q^4 \rightarrow (k-k')^4$

... Terms with m^2 and M^2 which we will neglect (ultra-rel. limit)

$$s = (k+p)^2 \simeq 2k\cdot p \simeq 2k'\cdot p'$$

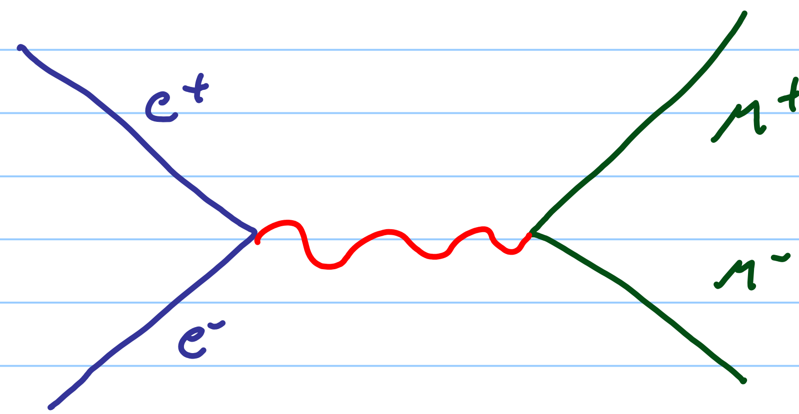
$$T = (k-k')^2 \simeq -2k\cdot k' \simeq -2p\cdot p'$$

$$u = (k-p')^2 \simeq -2k\cdot p' \simeq -2k'\cdot p$$

$$|M|^2 = 2e^4\left(\frac{s^2+u^2}{T^2}\right)$$

Cross Section Calculations (adding spin)

What about:



$$K' \leftrightarrow -p \quad , \quad s \longleftrightarrow t$$

$$|M|^2 = 2e^4\left[\frac{t^2 + u^2}{s^2}\right]$$

Using previous formulas:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} 2 \cdot e^4 \left[1/2 \left(1 + \cos^2\theta\right)\right] \quad , \quad \alpha = e^2/4\pi$$

integrate over θ, φ:

$$\boxed{\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}$$

Cross Section Calculations (adding spin)

The result we got: $\frac{4\pi\alpha^2}{3s}$ is in natural units

Inserting the missing $\hbar$ and c gives $\frac{4\pi(\alpha c\hbar)^2}{3s}$

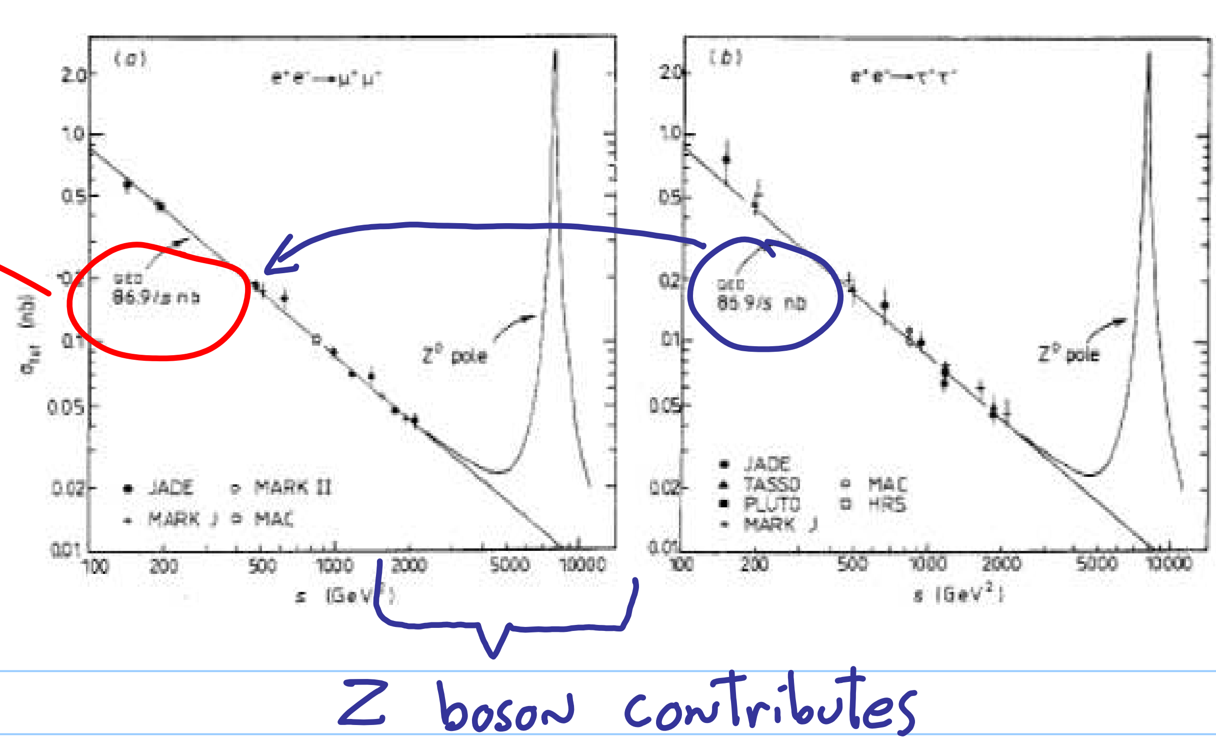
$\Rightarrow \sigma = \frac{x}{s}$ where $x = \frac{4}{3}\pi \cdot \frac{1}{137^2} \cdot (3\times10^8 \, m/s \cdot 6.6\times10^{-25}\, GeV\,s)^2$

$\Rightarrow x = 87.5 \times 10^{-37}\, m^2 \cdot GeV^2$ barn $= 10^{-28}\, m^2$

$x = 87.5\ nb \cdot GeV^2$

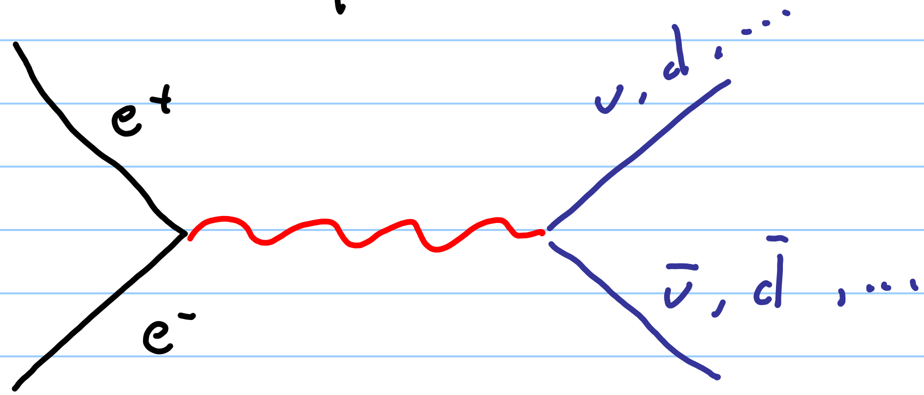
Pretty good!

- first order
- neglect masses
- approx. constants
- QED only



Cross Section Calculations (adding spin)

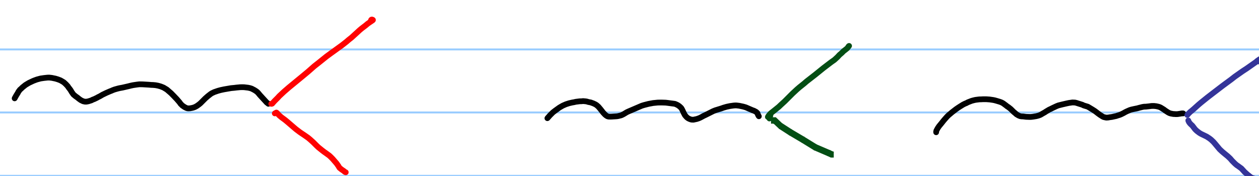
What about quarks?



$\frac{4\pi\alpha^2}{3s}$ needs to be modified for the charge of the quark:

$e \to e \cdot Q_q$ where $Q_q = 2/3$ for u, c
$Q_q = 1/3$ for d, s

# of colours i.e. $N_c = 3$



$\to \frac{4\pi\alpha^2}{s} Q_e^2$ E_{cm} needs to be $\gg 2m_q$

Cross Section Calculations (photon external lines)

Start from Maxwell's equations:

$$\Box^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu$$

We can chose a gauge such that $\partial_\mu A^\mu = 0$ (Lorentz gauge)

Maxwell's equations simplify to:

$$\Box^2 A^\nu = j^\nu$$

For a free photon we have: $\Box^2 A^\nu = 0$

The following can be a solution:

$A^\mu = \varepsilon^\mu(q)\, e^{-iq\cdot x}$ if: $q^2 = 0$

$\partial_\mu A^\mu = 0 \Rightarrow q_\mu \varepsilon^\mu = 0$

completeness relation:

$$\sum_T \varepsilon^{T*}_\mu \varepsilon^T_\nu = -g_{\mu\nu}$$

(real photons)

Cross Section Calculations (photon external lines)

Propagator for the electron:

$$-i(\not{p} - m)\psi = ie\gamma^\mu A_\mu \psi$$

$$\downarrow$$

$$\frac{1}{-i(\not{p}-m)} = \frac{i}{(\not{p}-m)} = \boxed{\frac{i(\not{p}+m)}{p^2-m^2}}$$

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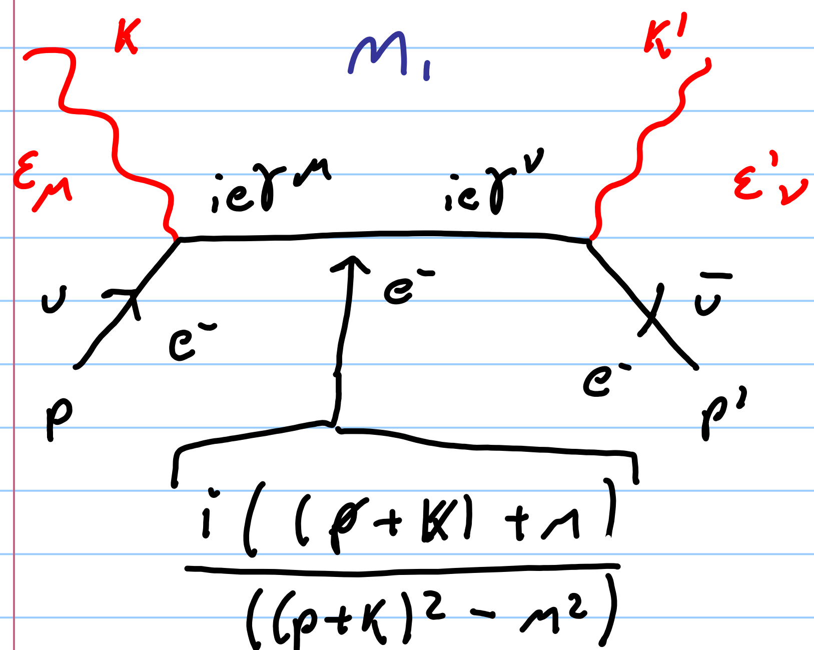
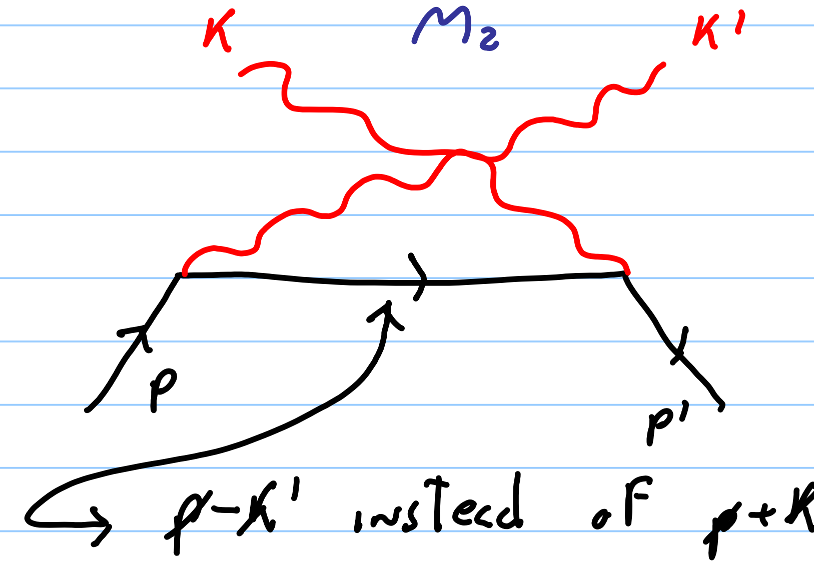
we had in T_{fi}: $\frac{1}{E_i - E_n}$, with $H_0|n\rangle = E_n|n\rangle$

we could write $\frac{1}{E_i - H_0}$ $\rightarrow$ $-i(E_i - H_0)\psi = -iV\psi$

$$\frac{i(\not{p}+m)}{p^2-m^2} = \frac{i\sum_s u\bar{u}}{p^2-m^2}$$

Cross Section Calculations (photon external lines)





$$-i\mathcal{M}_1 = \bar{v}^{(s')}(p')\left[\varepsilon'^{*}_\nu (ie\gamma^\nu)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(ie\gamma^\mu)\varepsilon_\mu\right]u^{(s)}(p)$$

$$-i\mathcal{M}_2 = \bar{v}^{(s')}(p')\left[\varepsilon_\mu (ie\gamma^\mu)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}(ie\gamma^\nu\varepsilon'^{*}_\nu\right]u^{(s)}(p)$$

We'll neglect m_e:

$$s = (k+p)^2 = 2k\cdot p = 2\cdot k'\cdot p'$$
$$t = (k-k')^2 = -2k\cdot k' = -2p\cdot p'$$
$$u = (k-p')^2 = -2k\cdot p' = -2p\cdot k'$$

Cross Section Calculations (photon external lines)

We have (neglecting electron mass):

$$\mathcal{M}_1 = \frac{\varepsilon'^{*}_{\nu}\, \varepsilon_\mu\, e^2\, \bar{u}(p')\, \gamma^\nu (\not{p} + \not{k}) \gamma^\mu\, v(p)}{s}$$

$$\mathcal{M}_2 = \frac{\varepsilon'^{*}_{\nu}\, \varepsilon_\mu\, \bar{u}(p')\, \gamma^\mu (\not{p} - \not{k}') \gamma^\nu\, v(p)}{u}$$

$$\sum_T \varepsilon^{*}_{\mu}\, \varepsilon^{T}_{\nu} = -g_{\mu\nu}$$

$$\overline{|\mathcal{M}_1|^2} = \frac{e^4}{4s^2} \sum_{s s'} \left(\bar{u}^{(s')} \gamma^\nu (\not{p} + \not{k}) \gamma^\mu v^{(s)}\right) \left(\bar{v}^{(s)} \gamma_\mu (\not{p} + \not{k}) \gamma_\nu u^{(s')}\right)$$

$$(\not{p} + m) = \not{p}$$

$$\overline{|\mathcal{M}_1|^2} = \frac{e^4}{4s^2} \mathrm{Tr}\left(\not{p}' \gamma^\nu (\not{p} + \not{k}) \gamma^\mu \not{p} \gamma_\mu (\not{p} + \not{k}) \gamma_\nu\right)$$

(annotations: $\not{p}' \gamma^\nu \to -2\not{p}'$, $\gamma^\mu \not{p} \gamma_\mu \to -2\not{p}$, $\sum_{s'} u\bar{u} \to \not{p}'$)

Cross Section Calculations (photon external lines)

$$\overline{|M_1|^2} = \frac{e^4}{s^2}\,\mathrm{Tr}\left(\not{p}'(\not{p}+\not{K})\not{p}(\not{p}+\not{K})\right)$$

$$\mathrm{Tr} = \underset{①}{\not{p}'\not{p}\not{p}\not{p}} + \underset{②}{\not{p}'\not{p}\not{p}\not{K}} + \underset{③}{\not{p}'\not{K}\not{p}\not{p}} + \underset{④}{\not{p}'\not{K}\not{p}\not{K}}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$$

Term ①, ②, ③ $= 0$ because $(p\cdot p) = 0$ since $m_e = 0$

$$\overline{|M_1|^2} = \frac{e^4}{s^2}\,\mathrm{Tr}(\not{p}'\not{K}\not{p}\not{K})$$

$$= \frac{4e^4}{s^2}\left[(p'\cdot K)(p\cdot K) - (p'\cdot p)\underbrace{(K\cdot K)}_{=0} + (p'\cdot K)(K\cdot p)\right]$$

$$= \frac{4e^4}{s^2}\,2(p'\cdot K)(p\cdot K)$$

$$= -2e^4\,\frac{u}{s}$$

$$s = (K+p)^2 = 2K\cdot p = 2\cdot K'\cdot p'$$
$$t = (K-K')^2 = -2K\cdot K' = -2p\cdot p'$$
$$u = (K-p')^2 = -2K\cdot p' = -2p\cdot K'$$

For $|M_2|^2$: $K \to -K'$ $u \longleftrightarrow s \Rightarrow -2e^4\,\dfrac{s}{u}$

$|M_1 M_2^*| = ?$

Cross Section Calculations (photon external lines)

$$\overline{M_1 M_2} =$$

$$\frac{e^4}{4su} \sum_{ss'} \left(\bar{v}^{(s')} \gamma^\nu (\not{p} + \not{k}) \gamma^\mu v^{(s)} \right) \left(\bar{v}^{(s)} \gamma_\nu (\not{p} - \not{k}') \gamma_\mu v^{(s')} \right)$$

$$|M_1 M_2| = \frac{e^4}{4su} \mathrm{Tr}\left(\not{p}' \gamma^\nu (\not{p} + \not{k}) \gamma^\mu \not{p} \gamma_\nu (\not{p} - \not{k}') \gamma_\mu \right)$$

$$\gamma^\nu \gamma^\mu \gamma_\nu = -2\gamma^\mu$$

$$\frac{e^4}{4su} \cdot -2 \, \mathrm{Tr}\left[\not{p}' \not{p} \gamma^\mu (\not{p} + \not{k})(\not{p} - \not{k}') \gamma_\mu \right]$$

$$= \frac{e^4}{4su} \cdot -2 \cdot -4 \, (p+k) \cdot (p-k') \, \mathrm{Tr} \, \not{p}' \not{p}$$

$$= 0 + \frac{u}{2} + \frac{s}{2} + \frac{t}{2}$$

$$\Rightarrow |M_1 M_2| = 0$$

$$s = (k+p)^2 = 2k \cdot p = 2 \cdot k' \cdot p'$$

$$t = (k-k')^2 = -2k \cdot k' = -2p \cdot p'$$

$$u = (k-p')^2 = -2k \cdot p' = -2p \cdot k'$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

Cross Section Calculations (photon external lines)

Pair Annihilation: $e^+ e^- \to \gamma\gamma$

we can obtain the result by crossing the amplitude for $e^-\gamma \to e^-\gamma$

$\longrightarrow$ replace $k, \varepsilon \to -k_2, \varepsilon_2^*$

$p' \to -p_1 \qquad \bar{u}^{(s')}(p') \to \bar{v}^{(s_1)}(p_1)$

$$\overline{|M|^2} = 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)$$